ON A FAMILY OF RIEMANNIAN MANIFOLDS DEFINED ON AN m-DISK

Dedicated to Professor MASARU OSIMA on his 60th birthday

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1. The Riemannian manifold O_n^m

Let R^m be the m-dimensional coordinate space with the canonical coordinates $u_1, u_2, \cdots, u_m$ and D^m be the unit m-disk

$$(u, u) := \sum_i u_i u_i < 1,$$

where $u=(u_1, \cdots u_m)$. We denote the Riemannian manifold defined on D^m with the following metric:

$$ds^2=(1-\sum_i u_i u_i)^{n-2}\{\sum_i du_i du_i - \sum_{i<j}(u_i du_j - u_j du_i)^2\} \tag{1.1}$$

by O_n^m, where n is a real constant.

In order to give a meaning of (1.1), suppose that n is an integer $\geqq 2$ and consider the unit $(n+m-1)$-sphere $S^{n+m-1}\subset R^{n+m}$ given by $\sum_{i=1}^{n+m} u_i u_i=1$. Let us consider as

$$R^{n+m}=R^n\times R^m$$

and take a smooth curve C in D^m. Then, for C we construct an n-dimensional submanifold $M^n(C)$ in S^{n+m-1} as follows:

$$M^n(C)=\{\cup S^{n-1}(\rho)\times u,\ u\in C\}, \tag{1.2}$$

where

$$\rho=\rho(u):=\sqrt{1-\sum_{i=1}^{m} u_i u_i} \tag{1.3}$$

and $S^{n-1}(\rho)$ is the $(n-1)$-sphere of radius ρ about the origin of R^n.

The n-dimensional volume of $M^n(C)$ is clearly given by the formula:

(1. 4) $$V(M^n(C)) = c_{n-1}\int_C \rho^{n-1}\sqrt{d\rho d\rho + (du, du)},$$

where c_{n-1} is the volume of the unit $(n-1)$-sphere S^{n-1}, i. e.

$$c_{n-1} = 2\pi^{n/2}/\Gamma(n/2).$$

Lemma 1. *The metric* (1. 1) *can be written as*

$$ds^2 = \rho^{2(n-1)}\{d\rho\, d\rho + (du, du)\}.$$

Proof. From $\rho^2 = 1-(u, u)$, we have $\rho d\rho = -(u, du)$. Hence

$$\rho^{2(n-1)}\{d\rho d\rho + (du, du)\} = \rho^{2(n-1)}\left\{\frac{(u, du)^2}{\rho^2} + (du, du)\right\}$$
$$= \rho^{2(n-2)}\{(u, du)^2 + (1-(u, u))(du, du)\}.$$
$$= \rho^{2(n-2)}\{(du, du) - ((u, u)(du, du) - (u, du)^2)\}$$
$$= (1 - \sum_i u_i u_i)^{n-2}\{\sum_i du_i du_i - \sum_{i<j}(u_i du_j - u_j du_i)^2\}$$

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Lemma 1 and (1. 4) imply immediately the following

Lemma 2. *An extremal of the volume of the family of the submanifolds* $\{M^n(C)$; *C is a smooth curve in* $D^m\}$ *in the* $(n+m-1)$*-sphere corresponds to a geodesic of* O_n^m *and vice versa.*

Remark. In the definition of O_n^m, we consider n as a real number. Especially, the cases of $n=1, 0$, have the following meanings:

O_1^m is the representation of the north hemisphere of S^m through the orthogonal projection onto the equatorial hyperplane of $R^{m+1}(\supset S^m)$.

O_0^m is the Cayley-Klein representation of the hyperbolic m-space of curvature 1. In fact, for any two points u, $v=u+du$ in D^m, let p, q be the points of intersection of the straight line joining u and v and the unit $(m-1)$-sphere $S^{m-1} = \partial D^m$. Denoting p and q in the form $(1-\lambda)u + \lambda v$, we have easily

$$(du, du)\lambda^2 + 2(u, du)\lambda - \rho^2 = 0,$$

hence

$$\lambda = \frac{-(u, du) \pm \delta_s}{(du, du)} := \lambda_\pm,$$

where

$$\delta_s^2 = (du, du) - \sum_{i<j}(u_i du_j - u_j du_i)^2.$$

Thus, we have the cross ratio of the four points u, v, p, q :

$$R(u, v: p, q)=\frac{\lambda_+}{1-\lambda_+}\cdot\frac{1-\lambda_-}{\lambda_-}=\frac{\rho^2-(u, du)+\delta_s}{\rho^2-(u, du)-\delta_s},$$

from which

$$\log R(u, v: p, q)=\log\left(1-\frac{(u, du)-\delta_s}{\rho^2}\right)-\log\left(1-\frac{(u, du)+\delta_s}{\rho^2}\right)$$
$$=\frac{2\delta_s}{\rho^2}+[2],$$

where [2] denotes the part of higher order of du, when we regard du as infinitesimal. Therefore, the Riemannian metric of the hyperbolic m-space H^m in this representation can be written as

$$ds^2=\frac{a\delta_s^2}{\rho^4}=a(1-(u, u))^{-2}\{(du, du)-\sum_{i<j}(u_i du_j-u_j du_i)^2\},$$

where a is a constant.

2. Geodesics of O_n^m

We shall investigate the geodesics of O_n^m.

From (1. 1), the components of the metric tensor of O_n^m are

$$g_{ij}=\rho^{2n-4}(\rho^2\delta_{ij}+u_i u_j) \tag{2. 1}$$

and

$$g^{ij}=\rho^{-2n+2}(\delta^{ij}-u^i u^j), \tag{2. 2}$$

where δ_{ij} are the Kronecker's δ and $u^i=u_i$. From (2. 1), we have

$$\frac{\partial g_{ij}}{\partial u^k}=\rho^{2n-6}\{\rho^2(u_i\delta_{jk}+u_j\delta_{ik})-2(n-1)\rho^2 u_k\delta_{ij}-2(n-2)u_i u_j u_k\}$$

and

$$\begin{aligned}[ij, k]:&=\frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial u^i}+\frac{\partial g_{ik}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^k}\right\}\\&=\rho^{2n-6}[\rho^2\{nu_k\,\delta_{ij}-(n-1)(u_i\delta_{jk}+u_j\delta_{ik})\}\\&\qquad-(n-2)u_i u_j u_k].\end{aligned} \tag{2. 3}$$

Thus, using the Einstein convention, the Christoffel's symbols of O_n^m in the coordinates u^i are given by

$$\begin{aligned}\left\{\begin{matrix} l \\ ij \end{matrix}\right\} &= g^{lk}[ij, k] \\ &= \rho^{-4}(\delta^{lk} - u^l u^k)\,[\rho^2\{nu_k\delta_{ij} - (n-1)(u_i\delta_{jk} + u_j\delta_{ik})\} \\ &\quad -(n-2)\,u_i u_j u_k] \\ &= \rho^{-4}[\rho^2\{nu^l\,\delta_{ij} - (n-1)\,(u_i\delta^l_j + u_j\delta^l_i)\} - (n-2)u_i u_j u^l \\ &\quad -\rho^2 u^l\{n(u, u)\delta_{ij} - 2(n-1)u_i u_j\} + (n-2)(u, u)u^l u_i u_j],\end{aligned}$$

i. e.

$$\left\{\begin{matrix} l \\ ij \end{matrix}\right\} = \frac{1}{\rho^2}\,[n(\rho^2\delta_{ij} + u_i u_j)u^l - (n-1)\,(u_i\delta^l_j + u_j\delta^l_i)]. \tag{2.4}$$

Theorem 1. *For any p-dimensional linear space $E^p(p<m)$ through the origin of R^m, $D^m \cap E^p$ is a totally geodesic submanifold of O^m_n, which is an O^p_n.*

Proof. As easily seen by Lemma 1, the metric (1. 1) is invariant under the rotations of R^m about the origin. Hence, we may suppose that E^p is given by

$$u_{p+1} = u_{p+2} = \cdots = u_m = 0.$$

For any tangent vector fields $X = \sum_{a=1}^{p} X^a \partial/\partial u^a$, $Y = \sum_{a=1}^{p} Y^a \partial/\partial u^a$ of $E^p \cap D^m$, we put

$$\nabla_X Y = \sum_{i=1}^{m} Z^i \partial/\partial u^i,$$

where ∇ denotes the covariant differentiation of O^m_n and Z^i is given by

$$Z^i = \sum_a \frac{\partial Y^i}{\partial u^a} X^a + \sum_{a,b} \left\{\begin{matrix} i \\ ab \end{matrix}\right\} Y^a X^b.$$

By means of (2. 4), on $E^p \cap D^m$ we have

$$\left\{\begin{matrix} i \\ ab \end{matrix}\right\} = -\frac{n-1}{\rho^2}(u_a\delta^i_b + u_b\delta^i_a) = 0$$

$$\text{for } i > p \text{ and } a, b \leqq p.$$

Hence we have

$$Z^i = 0 \qquad \text{for } i > p,$$

that is $\Delta_X Y$ is also a tangent vector field of $E^p \cap D^m$. This shows that

$E^p \cap D^m$ is a totally geodesic submanifold of O_n^m, which can be considered as an O_n^p by the induced metric from O_n^m. Q. E. D.

Corollary. *Any geodesic of O_n^m lies on a plane through the origin of R^m and can be considered as a geodesic of O_n^2.*

3. Certain properties of $M^n(C)$ in S^{n+m-1}

In this section, we suppose that n is an integer $\geqq 2$. By means of Lemma 2, an extremal of the volume of the family of the submanifolds $\{M^n(C)\}$ corresponds to a geodesic of O_n^m and then C is also a geodesic of an $O_n^2 \subset O_n^m$ by Corollary of Theorem 1. Accordingly, $M^n(C)$ can be considered as

$$M^n(C) \subset S^{n+1} \subset S^{n+m-1}$$

and it belongs to a family of hypersurfaces of S^{n+1}, which has two principal curvatures with multiplicity 1 and $n-1$.

Now, let C be a smooth curve in D^m not passing through the origin of D^m and $\bar{s}$ be its arclength. We take an orthonomal frame field $(q, \bar{\xi}_1, \bar{\xi}_2, \cdots \bar{\xi}_m)$ along C in R^m such that

$$q = f\bar{\xi}_1 - h\bar{\xi}_2 \quad (h \geqq 0), \tag{3.1}$$

where q also denotes the position vector of the moving point of C and

$$\bar{\xi}_1 = \frac{dq}{d\bar{s}}. \tag{3.2}$$

If q is not parallel to $\bar{\xi}_1$, $\bar{\xi}_2$ is determined uniquely at q. We have easily

$$1 - \rho^2 = f^2 + h^2, \tag{3.3}$$

where $\rho^2 = 1 - (q, q)$ by (1.3). From (3.2) and (3.3) we obtain

$$\rho \frac{d\rho}{d\bar{s}} = -\left(q, \frac{dq}{d\bar{s}}\right) = -(q, \bar{\xi}_1) = -f,$$

hence

$$\frac{d\rho}{d\bar{s}} = -\frac{f}{\rho}. \tag{3.4}$$

We put

$$\bar{k}_a := \left(\frac{d\bar{\xi}_2}{d\bar{s}}, \bar{\xi}_a\right), \quad a = 1, 3, \cdots, m. \tag{3.5}$$

Especially we have

$$\bar{k}_1 = \left(\frac{d\bar{\xi}_2}{d\bar{s}}, \bar{\xi}_1\right) = -\left(\frac{d\bar{\xi}_1}{d\bar{s}}, \bar{\xi}_2\right), \tag{3.6}$$

which shows that $-\bar{k}_1\bar{\xi}_2$ is the orthogonal projection of the principal curvature vector $\frac{d\bar{\xi}_1}{d\bar{s}}$ of C onto the plane through the origin of D^m and the tangent line of C at q.

On the other hand, let $(\bar{e}_1, \cdots \bar{e}_n)$ be the moving orthonormal frame of R^n at the origin and put

$$d\bar{e}_i = \sum_j \bar{\omega}_{ij}\bar{e}_j, \qquad \bar{\omega}_{ij} + \bar{\omega}_{ji} = 0. \tag{3.7}$$

The generating moving point p of $M^n(C)$ is given by

$$p = q + \rho\bar{e}_n = \rho\bar{e}_n + f\bar{\xi}_1 - h\bar{\xi}_2, \tag{3.8}$$

from which we obtain by differentiation

$$dp = \rho\sum_{a=1}^{n-1}\bar{\omega}_{na}\bar{e}_a + d\bar{s}\left(\frac{d\rho}{d\bar{s}}\bar{e}_n + \bar{\xi}_1\right).$$

Using (3.3) and (3.4), if we put

$$e_a = \bar{e}_a, \quad e_n = \frac{-f\bar{e}_n + \rho\bar{\xi}_1}{\sqrt{1-h^2}}, \qquad \omega_a = \rho\bar{\omega}_{na}, \quad \omega_n = \frac{\sqrt{1-h^2}}{\rho}d\bar{s}, \tag{3.9}$$

then we have the equality

$$dp = \sum_{i=1}^{n}\omega_i e_i$$

and $(p, e_1, \cdots, e_n)$ is an orthonormal frame of $M^n(C)$ at p.

Next, if we put

$$e_{n+1} = -\frac{h}{\sqrt{1-h^2}}(\rho\bar{e}_n + f\bar{\xi}_1) - \sqrt{1-h^2}\,\bar{\xi}_2, \tag{3.10}$$

then

$$\|e_{n+1}\|^2 = \frac{h^2}{1-h^2}(\rho^2 + f^2) + 1 - h^2 = 1.$$

e_{n+1} is clearly orthogonal to $e_1, e_2, \cdots, e_n$. Using (3. 3) and (3. 8), we obtain

$$\begin{aligned}(p, e_{n+1}) &= (\rho\bar{e}_n + f\bar{\xi}_1 - h\bar{\xi}_2, \ e_{n+1}) \\ &= -\frac{h\rho^2}{\sqrt{1-h^2}} - \frac{hf^2}{\sqrt{1-h^2}} + h\sqrt{1-h^2} = 0,\end{aligned}$$

which shows that e_{n+1} is also tangent to S^{n+m-1}.

Furthermore, putting

(3. 11) $$e_\lambda = \bar{\xi}_{\lambda-n+1}, \qquad \lambda > n+1,$$

we obtain a moving orthonormal frame $(p, e_1, \cdots, e_{n+m-1})$ of S_{n+m-1} defined along $M'(C)$. From this frame, we obtain by the covariant differentiation D on S^{n+m-1} the following:

$$\begin{aligned}\omega_{a,n+1} &= (De_a, \ e_{n+1}) = (d\bar{e}_a, \ e_{n+1}) \\ &= (d\bar{e}_a, \ -\frac{h}{\sqrt{1-h^2}}(\rho\bar{e}_n + f\bar{\xi}_1) - \sqrt{1-h^2}\,\bar{\xi}_2) \\ &= \frac{h\rho\bar{\omega}_{na}}{\sqrt{1-h^2}} = \frac{h}{\sqrt{1-h^2}}\omega_a \qquad \text{for } a = 1, 2, \cdots, n-1\end{aligned}$$

and

$$\begin{aligned}\omega_{n,n+1} &= (De_n, \ e_{n+1}) = (de_n, \ e_{n+1}) \\ &= \left(d\frac{-f\bar{e}_n + \rho\bar{\xi}_1}{\sqrt{1-h^2}}, \ e_{n+1}\right) = \frac{1}{\sqrt{1-h^2}}(d(-f\bar{e}_n + \rho\bar{\xi}_1), \ e_{n+1}).\end{aligned}$$

Since

$$\begin{aligned}-\sqrt{1-h^2}\,\omega_{n,n+1} &= \left(\left(-\frac{df}{d\bar{s}}\bar{e}_n + \frac{d\rho}{d\bar{s}}\bar{\xi}_1 + \rho\frac{d\bar{\xi}_1}{d\bar{s}}\right)d\bar{s} - fd\bar{e}_n,\right. \\ &\qquad \left.\frac{h}{\sqrt{1-h^2}}(\rho\,\bar{e}_n + f\bar{\xi}_1) + \sqrt{1-h^2}\,\bar{\xi}_2\right) \\ &= \left\{\frac{h}{\sqrt{1-h^2}}\left(-\rho\frac{df}{d\bar{s}} + f\frac{d\rho}{d\bar{s}}\right) - \rho\sqrt{1-h^2}\,{}_1\bar{k}\right\}d\bar{s},\end{aligned}$$

using (3. 4) and (3. 9) we have

$$\omega_{n,n+1}=\left\{\frac{h}{\sqrt{(1-h^2)^3}}\left(f^2+\rho^2\frac{df}{d\bar{s}}\right)+\frac{\bar{k}_1\rho^2}{\sqrt{1-h^2}}\right\}\omega_n.$$

On the other hand, from (3. 1), (3. 2) and (3. 6) we obtain

$$\xi_1=\frac{dq}{d\bar{s}}=\frac{df}{d\bar{s}}\bar{\xi}_1+f\frac{d\bar{\xi}_1}{d\bar{s}}-\frac{dh}{d\bar{s}}\bar{\xi}_2-h\frac{d\bar{\xi}_2}{d\bar{s}},$$

which implies

$$1=\frac{df}{d\bar{s}}-h\left(\bar{\xi}_1,\ \frac{d\bar{\xi}_2}{d\bar{s}}\right)=\frac{df}{d\bar{s}}-h\bar{k}_1,$$

i. e.

(3. 12) $$\frac{df}{d\bar{s}}=1+h\bar{k}_1.$$

Taking the inner product of the above equality with $\bar{\xi}_2$, we obtain easily

$$0=f\left(\bar{\xi}_2,\ \frac{d\bar{\xi}_1}{d\bar{s}}\right)-\frac{dh}{d\bar{s}}=-\bar{k}_1f-\frac{dh}{d\bar{s}},$$

i. e.

(3. 13) $$\frac{dh}{d\bar{s}}=-\bar{k}_1f.$$

We obtain analogously the following:

(3. 14) $$f\left(\frac{d\bar{\xi}_1}{d\bar{s}},\ \bar{\xi}_a\right)=h\bar{k}_a,\qquad a=3,\ 4,\cdots,\ n-1.$$

Using (3. 12) and (3. 3), we have

$$\frac{h}{\sqrt{(1-h^2)^3}}\left(f^2+\rho^2\frac{df}{d\bar{s}}\right)+\frac{\bar{k}_1\rho^2}{\sqrt{1-h^2}}=\frac{h}{\sqrt{1-h^2}}+\frac{\bar{k}_1\rho^2}{\sqrt{(1-h^2)^3}}.$$

Hence, we obtain the following:

(3. 15) $$\begin{cases}\omega_{a,n+1}=\dfrac{h}{\sqrt{1-h^2}}\omega_a, & a=1,\ 2,\cdots,\ n-1;\\ \omega_{n,n+1}=\left(\dfrac{h}{\sqrt{1-h^2}}+\dfrac{\bar{k}_1\rho^2}{\sqrt{(1-h^2)^3}}\right)\omega_n. & \end{cases}$$

Then, for $\lambda>n+1$, by (3. 14) and (3. 9) we have

$$\omega_{a\lambda} = (De_a,\ e_\lambda) = (d\bar{e}_a,\ \bar{\xi}_{\lambda-n+1}) = 0 \ \text{ for } \ 1 \leqq a \leqq n-1,$$

$$\begin{aligned}\omega_{n\lambda} = (De_n,\ e_\lambda) &= (de_n,\ \bar{\xi}_{\lambda-n+1}) \\ &= \frac{1}{\sqrt{1-h^2}}(d(-f\bar{e}_n + \rho\bar{\xi}_1),\ \bar{\xi}_{\lambda-n+1}) \\ &= \frac{\rho}{\sqrt{1-h^2}}\left(\frac{d\bar{\xi}_1}{d\bar{s}},\ \bar{\xi}_{\lambda-n+1}\right)d\bar{s} = \frac{\rho}{\sqrt{1-h^2}}\cdot\frac{h}{f}\bar{k}_{\lambda-n+1}\cdot\frac{\rho}{\sqrt{1-h^2}}\omega_n \\ &= \frac{h\rho^2}{f(1-h^2)}\bar{k}_{\lambda-n+1}\,\omega_n,\end{aligned}$$

i. e.

$$\begin{cases}\omega_{a\lambda} = 0, \ a = 1, 2, \cdots, n-1; \\ \omega_{n\lambda} = \dfrac{h\rho^2}{f(1-h^2)}\bar{k}_{\lambda-n+1}\,\omega_n. \end{cases} \tag{3. 16}$$

Now we state the definition of principal normal vectors for a submanifold introduced by the author in [5]. In general, let M be a submanifold of a Riemannian manifold $\overline{M}$. A normal vector v at a point $x \in M$ is called a *principal normal vector* of M at x, if it satisfies the following condition:

There exists a tangent vector $u \in M_x$, $u \neq 0$, such that

$$T_u z = (u,\ z)\,v \qquad \text{for all } z \in M_x,$$

where M_x denotes the tangent space of M at x and T is the shape operator of M in $\overline{M}$. u is called a *principal tangent vector* for v.

It is evident that all the principal tangent vectors for v and the zero vector span a linear tangent subspace, which we denote by $E(x,\ v)$.

A C^∞ normal vector field V of M is called a *regular principal normal vector field*, if V is a principal normal vector at each point x of M and dim $E(x,\ V(x))$ is constant. When $\overline{M}$ is of constant curvature, $E(M,\ V) = \bigcup_x E(x,\ V(x))$ is a complete distribution of M (Theorem 1, [5]).

Now, going back to the previous situation, by means of (3. 15) and (3. 16), the shape operator of $M^n(C)$ as a submanifold of S^{n+m-1} can be written as follows: For any tangent vectors $X = \sum_i X_i e_i$, $Z = \sum_i Z_i e_i$,

$$\begin{aligned}T_X Z = &\left\{\frac{h}{\sqrt{1-h^2}}\sum_{a=1}^{n-1} X_a Z_a + \left(\frac{h}{\sqrt{1-h^2}} + \frac{\bar{k}_1\rho^2}{\sqrt{(1-h^2)^3}}\right)X_n Z_n\right\} e_{n+1} \\ &+ \frac{h\rho^2 X_n Z_n}{f(1-h^2)}\sum_{\lambda>n+1}\bar{k}_{\lambda-n+1}e_\lambda.\end{aligned} \tag{3. 17}$$

The formula (3. 17) implies immediately the following

Theorem 2. *$M = M^n(C)$ has two regular principal normal vector fields V and W given by*

$$V = \frac{h}{\sqrt{1-h^2}} e_{n+1},$$
$$W = \left(\frac{h}{\sqrt{1-h^2}} + \frac{k_1 \rho^2}{\sqrt{(1-h^2)^3}}\right) e_{n+1} + \frac{h\rho^2}{f(1-h^2)} \sum_{\lambda > n+1} \bar{k}_{\lambda-n+1} e_\lambda,$$

if the tangent lines of C do not pass through the origin of D^m. Then $E(M, V)$ and $E(M, W)$ are distributions of dimension $n-1$ and 1, respectively and

$$E(M, V) \oplus E(M, W) = T(M).$$

On the distributions in a sphere as V and W, we have the following theorem (Theorem 6, [6]):

Theorem 3. *Let $M^n (n \geqq 3)$ be a minimal submanifold of $S^{n+p} \subset R^{n-p+1}$ with two regular principal normal vector fields V and W such that*

$$E(M^n, V) \oplus E(M^n, W) = T(M^n).$$

Then, there exists an $(n+2)$-dimensional subspace E^{n+2} of R^{n+p+1} through the origin such that

$$M^n \subset E^{n+2} \cap S^{n+p}.$$

Theorem 4. *$M^n(C)$ is minimal in S^{n+m-1} if and only if C is a geodesic of O^m_n.*

Proof. By Theorem 2, $M^n(C)$ has two regular principal normal vector fields V and W satisfying the condition as in Theorem 3.

If $M^n(C)$ is minimal in S^{n+m-1}, by Theorem 3 there exists an $(n+2)$-dimensional linear subspace E^{n+2} of R^{n+m} through the origin such that

$$M^n(C) \subset E^{n+2} \cap S^{n+m-1}.$$

Hence, by the way of construction of $M^n(C)$, C must lie in a plane through the origin. Accordingly, by (3. 1), (3. 2) and (3. 5) we have

$$\bar{k}_3 = \bar{k}_4 = \cdots = \bar{k}_m = 0. \tag{3. 18}$$

Thus, the condition that $M^n(C)$ is minimal becomes

$$\frac{nh}{\sqrt{1-h_2}}+\frac{\rho^2\bar{k}_1}{\sqrt{(1-h^2)^3}}=0 \tag{3.19}$$

by means of (3.17). Hence, by a result in [4], C is a geodesic of O_n^2. By means of Corollary of Theorem 1, C is also a geodesic of O_n^m.

Conversely, if C is a geodesic of O_n^m, then it lies in a plane through the origin. Thus, (3.18) is true for C. By means of (3.17), $M^n(C)$ is minimal if (3.19) is true. Using the direction angle t of C in the plane, we have

$$\bar{k}_1=-1\Big/\left(h+\frac{d^2h}{dt^2}\right) \quad \text{and} \quad f=\frac{dh}{dt}.$$

Therefore, (3.19) can be written as

$$nh(1-h^2)\frac{d^2h}{dt^2}+\left(\frac{dh}{dt}\right)^2+(1-h^2)(nh^2-1)=0. \tag{3.20}$$

This is also a condition that C is a geodesic of O_n^2 (Proposition 1, [8]). Hence $M^n(C)$ must be minimal in S^{n+m-1}. Q.E.D.

Remark. In order to prove Theorem 4, we can use Lemma 2. But, we heve to take care of the following fact. If $M^n(C_0)$ is minimal, we may consider C_0 is a smooth arc in D^m and it is extremal with respect to the n-dimensional volume of the family of $M^n(C)$ such that C are smooth curves in D^m with the same nd points of C_0.

Finally, we shall give a remark on the representation of O_n^m like the Poincaré one of $H^2=O_0^2$.

Let us suppose that n is any real number and denote the line element (1.2) of O_n^m by

$$ds_n^2=(1-(u,u))^{n-2}[(1-(u,u))(du,du)+(u,du)^2]. \tag{3.21}$$

In D^m we take the change of coordinate system: $u=(u_1,\cdots,u_m)\longrightarrow x=(x_1,\cdots,x_m)$ given by

$$u=\frac{2x}{1+r^2}, \qquad r=\sqrt{(x,x)}. \tag{3.22}$$

Then, we have

$$1-(u,u)=\left(\frac{1-r^2}{1+r^2}\right)^2,$$

$$(du,\ du) = \frac{4}{(1+r^2)^4}\{(1+r^2)^2(dx,\ dx) - 4(x,\ dx)^2\},$$

$$(u,\ du) = \frac{4(1-r^2)}{(1+r^2)^3}(x,\ dx).$$

Substituting these into (3. 21), we obtain

$$ds_n^2 = \frac{4(1-r^2)^{2(n-1)}}{(1+r^2)^{2n}}(dx,\ dx),$$

i. e.

$$ds_n^2 = \frac{4(1-\sum_i x_i x_i)^{2(n-1)}}{(1+\sum_i x_i x_i)^{2n}} \sum_j dx_j dx_j. \tag{3. 23}$$

Especially, we have

$$ds_0^2 = \frac{4}{(1-\sum_i x_i x_i)^2} \sum_i dx_j dx_j, \tag{3. 24}$$

which is the Poincaré representation of the hyperbolic plane of curvature -1. Hence, we have from (3. 23) and (3. 24)

$$ds_n^2 = \left(\frac{1-\sum_i x_i x_i}{1+\sum_i x_i x_i}\right)^{2n} ds_0^2.$$

Therefore, we may call the expression (3. 24) the Poincaré representation of O_n^m.

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