K_O-COHOMOLOGIES OF THE DOLD MANIFOLDS

Dedicated to Professor MASARU OSIMA on the occasion of his sixtieth birthday

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Introduction

The purpose of this paper is to determine the K_O-cohomologies of the Dold manifold $D(m, n)$. As for the K_U-cohomologies of $D(m, n)$, one of the authors has determined in [6] and [7]. We inherit all the notations of [6] and [7]. We use K or KO instead of K_U or K_O.

Let $\pi: D(m, n) \longrightarrow D(m, n)/D(m, 0)$ be the projection and $\tilde{K}_A^{-i}(m, n) = \pi^! \tilde{K}_A^{-i}(D(m, n)/D(m, 0))$, where $A=O$ or U. Then, we have the following

Theorem 1.

$$\tilde{K}_A^{-i}(D(m, n)) = \tilde{K}_A^{-i}(m, n) + p^! \tilde{K}_A^{-i}(RP(m)),$$

where $A=O$ or U and $p: D(m, n) \longrightarrow RP(m)$ is the natural projection.

By this theorem, it is sufficient to calculate the summand $\tilde{K}O^{-i}(m, n)$ for our purpose, because $\tilde{K}O^{-i}(RP(m))$ is known in [8].

In [7, Proposition 2], we have the following two homeomorphisms:

(i) $h_1: D(m, n)/D(m-1, n) \approx S^m \wedge CP(n)^+$,

(ii) $h_2: D(m, n)/D(m, n-1) \approx S^n \wedge (RP(m+n)/RP(n-1))$,

which are basic in our method.

For the first time, we deal with $\tilde{K}O^{-i}(m, n)$ for $n=2r$, by induction on m with considering the exact sequence of the pair $(D(m, n), D(m-1, n))$. Here, the homeomorphism h_1 of (i) and the Bott sequence play important roles in our computations.

In case of $n=2r+1$, we can define algebraically a splitting homomorphism κ by using the results on $\tilde{K}O^{-i}(m, 2r)$ and obtain a splitting exact sequence

$$0 \longrightarrow \tilde{K}O^{-i}(D(m, 2r+1)/D(m, 2r)) \longrightarrow \tilde{K}O^{-i}(m, 2r+1) \underset{\kappa}{\rightleftarrows} \tilde{K}O^{-i}(m, 2r) \longrightarrow 0.$$

Therefore we have the following

Theorem 2.

$$\widetilde{KO}^{-i}(m, 2r+1) \cong \widetilde{KO}^{-i}(m, 2r) + \widetilde{KO}^{-i}(D(m, 2r+1)/D(m, 2r)).$$

In this direct sum decomposition, we have an isomorphism

$$\widetilde{KO}^{-i}(D(m, 2r+1)/D(m, 2r)) \cong \widetilde{KO}^{-i}(S^{2r+1} \wedge (RP(2r+1+m)/RP(2r)))$$

by the homeomorphism h_2 of (ii), and the right hand side is known in [9].

The results of $\widetilde{KO}^*(m, 2r)$ are stated as follows, where $\alpha_0 \in \widetilde{KO}^0(m, 2r)$ is the element defined in [6] (cf. §3) and w and z are generators of $KO^{-1}(point)$ and $KO^{-4}(point)$ respectively.

Theorem 3. *$\widetilde{KO}^*(m, 2r)$ is a graded abelian group generated by the following elements:*

Case $m \geqq 3$

free basis: $\alpha_0, \cdots, \alpha_0^r, \ s, s\alpha_0, \cdots, s\alpha_0^{r-1},$
$z\alpha_0, \cdots, z\alpha_0^r, zs, zs\alpha_0, \cdots\cdots, zs\alpha_0^{r-1},$

generators of order 2: $w\alpha_0, \cdots, w\alpha_0^r, ws, ws\alpha_0, \cdots ws\alpha_0^{r-1},$
$w^2\alpha_0, \cdots, w^2\alpha_0^r, w^2s, w^2s\alpha_0, \cdots w^2s\alpha_0^{r-1},$

where s *is an element in* $\widetilde{KO}^{6-m}(D(m, 2r))$.

Case $m=1$

free basis: $\alpha_0, \cdots, \alpha_0^r, a, a\alpha_0, \cdots\cdots, a\alpha_0^{r-1},$
$\gamma_3, \gamma_3\alpha_0, \cdots \gamma_3\alpha_0^{r-1}, \gamma_7, \gamma_7\alpha_0, \cdots, \gamma_7\alpha_0^{r-1},$

generators of order 2: $w\alpha_0, \cdots, w\alpha_0^r, wa, wa\alpha_0, \cdots, wa\alpha_0^{r-1},$

where a *is an element in* $\widetilde{KO}^{-4}(D(1, 2r))$ *such that* $z\alpha_0 = 2a$ *and* γ_i *is an element in* $\widetilde{KO}^{-i}(D(1, 2r))$ *for* $i=3, 7$.

Case $m=2$

free basis: $\alpha_0, \cdots, \alpha_0^r, b, b\alpha_0, \cdots, b\alpha_0^{r-1},$
$\gamma_0, \gamma_0\alpha_0, \cdots, \gamma_0\alpha_0^{r-1}, \gamma_4, \gamma_4\alpha_0, \cdots, \gamma_4\alpha_0^{r-1},$

generators of order 2: $w\alpha_0, \cdots, w\alpha_0^r, wb, wb\alpha_0, \cdots, wb\alpha_0^{r-1},$
$w^2\alpha_0, \cdots, w^2\alpha_0^r, w^2b, w^2b\alpha_0, \cdots, w^2b\alpha_0^{r-1},$

where b *is an element in* $\widetilde{KO}^{-4}(D(2, 2r))$ *such that* $z\alpha_0 = 2b$ *and* γ_i is *an element in* $\widetilde{KO}^{-i}(D(2, 2r))$ *for* $i=0, 4$.

Since $KO^*(point)$ is a graded ring with unit 1 generated by w and z with the relations $2w=0$, $w^3=0$, $wz=0$ and $z^2=4$, we can restate the above theorem for $m \geqq 3$ as follows.

Theorem 4. *In case of* $m \geqq 3$, $\widetilde{KO}^*(m, 2r)$ *is a graded* $KO^*(point)$-*free module with basis* $\alpha_0, \cdots, \alpha_0^r, s, s\alpha_0, \cdots, s\alpha_0^{r-1}$, *where degree* $\alpha_0 = 0$ *and degree* $s = 6 - m$.

We state on the results of $\widetilde{KO}^0(D(m, n))$ in detail, namely

Theorem 5.

1) $p^!\widetilde{KO}^0(RP(m)) = Z_{2^f}$, *which is generated by* λ_0 (*cf.* §3) *with two relations* $\lambda_0^2 = -2\lambda_0$ *and* $\lambda_0^{f+1} = 0$, *where* $f = \varphi(m)$ *is the number of integers* q *such that* $0 < q \leqq m$ *and* $q \equiv 0, 1, 2$ *or* $4 \bmod 8$.

2) *Case* $m = 8t$, $8t+1$, $8t+3$ *or* $8t+7$.

$\widetilde{KO}^0(m, 2r) = Z^{(r)}$, *which is generated by* $\alpha_0, \cdots, \alpha_0^r$.[1]

Case $m = 8t+2$ *or* $8t+6$.

$\widetilde{KO}^0(m, 2r) = Z^{(2r)}$, *which is generated by* $\alpha_0, \cdots, \alpha_0^r, \zeta, \zeta\alpha_0, \cdots, \zeta\alpha_0^{r-1}$, *where* $\zeta = s$ *if* $m = 8t+6$, $\zeta = zs$ *if* $m = 8t+2$ $(t > 0)$ *and* $\zeta = \gamma_0$ *if* $m = 2$.

Case $m = 8t+4$ *or* $8t+5$.

$\widetilde{KO}^0(m, 2r) = Z^{(r)} + Z_2^{(r)}$, *whose free part is generated by* $\alpha_0, \cdots, \alpha_0^r$ *and torsion part is generated by* $\theta, \theta\alpha_0, \cdots, \theta\alpha_0^{r-1}$, *where* $\theta = ws$ *if* $m = 8t+5$ *and* $\theta = w^2 s$ *if* $m = 8t+4$.

3) *The groups* $\widetilde{KO}^0(D(m, 2r+1)/D(m, 2r))$ *are isomorphic to the following groups*:

r \ m	$8t$	$8t+1$	$8t+2$	$8t+3$	$8t+4$	$8t+5$	$8t+6$	$8t+7$
even	Z_2	Z_2	$Z+Z_2$	Z_2	Z_2	Z_2	$Z+Z_2$	Z_2
(generators)	α_0^{r+1}	α_0^{r+1}	$\zeta\alpha_0^r, \alpha_0^{r+1}$	α_0^{r+1}	α_0^{r+1}	α_0^{r+1}	$\zeta\alpha_0^r, \alpha_0^{r+1}$	α_0^{r+1}
odd	0	0	Z	Z_2	Z_2+Z_2	Z_2	Z	0
(generators)			ζ'	y	$x, \theta\alpha_0^r$	$\theta\alpha_0^r$	$\zeta\alpha_0^r$	

where $2\zeta' = \zeta\alpha_0^r$.

As for the ring structures of $\widetilde{KO}^0(D(m, n))$ we have the following

Theorem 6. *As for multiplicative structures of* $\widetilde{KO}^0(D(m, n))$ *we have the following relations*:

1) $\lambda_0^2 = -2\lambda_0$, $\lambda_0^{f+1} = 0$, $\lambda_0\alpha_0 = 0$.

2) $\alpha_0^{r+1} = 0$ *if* $n \not\equiv 1 \bmod 4$; $2\alpha_0^{r+1} = \alpha_0^{r+2} = 0$ *if* $n \equiv 1 \bmod 4$.

1) $G^{(r)}$ means the direct sum $G + \cdots + G$ (r-copies).

3) $\zeta\alpha_0^r=0$ *if* n *is even*; $\zeta\alpha_0^{r+1}=0$ *if* n *is odd*; $\lambda_0\zeta=\zeta^2=0$.
4) $\theta\alpha_0^r=0$ *if* $m\not\equiv 3 \bmod 4$; $\theta\alpha_0^{r+1}=0$ *if* $n\equiv 3 \bmod 4$; $\lambda_0\theta=\theta^2=0$.
5) $x^2=0$ *or* $\theta\alpha_0^r$; $\lambda_0 x=0$, x, $\theta\alpha_0^r$ *or* $x+\theta\alpha_0^r$; $x\alpha_0=x\theta=0$.
6) $y^2=0$; $\lambda_0 y=0$ *or* y; $y\alpha_0=0$.

Theorem 1 is proved in § 1. After some preparations on abelian groups in § 2 and on $\widetilde{K}^*(D(m, n))$ in § 3, we prove Theorem 3 in §§4—9. We determine the rank of $\widetilde{KO}^{-i}(m, 2r)$ (Proposition (4. 8)) in §4, and investigate the homomorphisms in the Bott sequence (Lemma (5. 2)) in § 5. Theorem 3 for $m=1, 2$ and 3 are proved in § 6, using the fact that $\widetilde{KO}^{-3}(D(3, 2r))=0$ which is proved in § 7. The general inductive proof of Theorem 3 is done in § 8 by the routine calculations. We change some generators of $\widetilde{KO}^{-i}(D(m, 2r))$ in § 9. Theorem 2 is proved in § 10 and Theorems 5 and 6 in § 11.

1. Direct summand

1. 1. Proof of Theorem 1. It is easy to see $D(m, 0)\approx RP(m)$. Under this identification, consider the following exact sequence

$$(1.1)\quad \longrightarrow \widetilde{K}_A^{-i}(D(m, n)/D(m, 0)) \xrightarrow{\pi!} \widetilde{K}_A^{-i}(D(m, n)) \underset{p!}{\overset{i!}{\rightleftarrows}} \widetilde{K}_A^{-i}(D(m, 0)) \longrightarrow,$$

where p: $D(m, n) \longrightarrow RP(m)$ is the natural projection, i: $RP(m) \longrightarrow D(m, n)$ is the inclusion defined by $i([x_0, \cdots, x_m])=[x_0, \cdots, x_m, 1, 0, \cdots, 0]$ and π: $D(m, n) \longrightarrow D(m, n)/D(m, 0)$ is the projection. Here, $i^! p^! =$ identity, then we have the theorem.

1. 2. Commutativity of the following diagram

$$\begin{array}{ccc} D(m, n)/D(m-1, n) & \approx & S^m \wedge CP(n)^+ \\ \bar{p}\downarrow\ \uparrow\bar{i} & & \uparrow i \\ RP(m)/RP(m-1) & \approx & S^m \wedge CP(0)^+ \end{array}$$

implies that we may identify $\widetilde{K}_A^{-i}(S^m \wedge CP(0)^+)$ with the summand $\widetilde{K}_A^{-i}(S^m)$ of $\widetilde{K}_A^{-i}(S^m \wedge CP(n)^+)=\widetilde{K}_A^{-i}(S^m \wedge CP(n))+\widetilde{K}_A^{-i}(S^m)$. Then we have the following long exact sequence

$$(1.2)\quad \to\widetilde{K}_A^{-i}(S^m \wedge CP(n))\xrightarrow{f!}\widetilde{K}_A^{-i}(m, n)\xrightarrow{i!}\widetilde{K}_A^{-i}(m-1, n)\xrightarrow{\delta}\widetilde{K}_A^{-i+1}(S^m \wedge CP(n)),$$

where $f=h_1\pi$ and h_1 is the homeomosphism of (i) in the introduction, and δ is the boundary operation in K_A-cohomology theory. (1. 2) is a direct summand of the long exact sequence of the pair $(D(m, n), D(m-1, n))$.

Theorem 1 and (1. 2) are also true, when K_A^* is replaced by an arbitrary cohomology theory.

2. Preparations on abelian groups

Let $Z^{(r)}$ denote a free abelian group of rank r, let $Z_2^{(s)}$ denote an abelian group which is the direct sum of s cyclic groups of order 2, and let $<a_1, \cdots, a_n>$ denote the free abelian group generated by $a_1, \cdots, a_n$. Then we have the following two lemmas which are useful for the computation of $\widetilde{KO}^{-i}(D(m, 2r))$.

Lemma (2.1). *Let* $0 \longrightarrow Z^{(r)} \xrightarrow{\kappa} A \xrightarrow{\sigma} Z_2^{(s)} \longrightarrow 0$ *be an exact sequence and* A *be an abelian group which contains* $Z_2^{(s)}$ *as a subgroup. Then* A *is isomorphic to* $Z^{(r)}+Z_2^{(s)}$.

Proof. Let B be the subgroup $Z_2^{(s)}$ of A. Since Im κ is free, we have $B \cap \text{Im}\ \kappa = 0$, and so $\sigma|_B : B \longrightarrow Z_2^{(s)}$ is monomorphic. This shows that $\sigma|_B$ is isomorphic and the lemma follows.

In virtue of the fundamental theorem of abelian group, we can easily see the following :

Lemma (2.2). *Let* $0 \longrightarrow Z^{(s)} \xrightarrow{\kappa} A \longrightarrow Z^{(r)}+Z_2^{(s)} \longrightarrow 0$ *be an exact sequence and* A *be a free abelian group of rank* $r+s$. *Then, for any basis* $e_1, \cdots, e_s$ *of* $Z^{(s)}$ *we can choose a basis* $u_1, \cdots, u_{r+s}$ *of* A *such that* $\kappa(e_i) = 2u_i$ $(1 \leqq i \leqq s)$.

3. Known results on $\widetilde{K}^*(D(m, 2r))$

We recall from [6] the results on $\widetilde{K}^*(D(m, 2r))$ which is needed for the computation of $\widetilde{KO}^*(D(m, n))$. Denote by ξ the canonical real line bundle over the real projective m-space $RP(m)$, and $\xi_1 = p^! \xi$ the induced bundle of ξ by the projection $p : D(m, n) \longrightarrow RP(m)$; by η the canonical complex line bundle over the complex projective n-space $CP(n)$; and denote by η_1 the canonical real 2-plane bundle over $D(m, n)$ (cf. [6, § 2]). Then the generators for our groups are defined as follows :

$$\lambda = \xi - 1 \quad \in \widetilde{KO}^0(RP(m)),$$
$$\nu = \varepsilon\lambda \quad \in \widetilde{K}^0(RP(m)),$$
$$\mu = \eta - 1 \quad \in \widetilde{K}^0(CP(n)),$$
$$\mu_0 = \rho\mu \quad \in \widetilde{KO}^0(CP(n)),$$

$$\begin{aligned}
\mu_i &= \rho g^i \mu && \in \widetilde{KO}^{-2i}(CP(n)) \quad (i=1,2,3),\\
\alpha_0 &= \gamma_1 - \xi_1 - 1 && \in \widetilde{KO}^0(D(m,n)),\\
\alpha &= \varepsilon\alpha_0 && \in \widetilde{K}^0(D(m,n)),\\
\gamma &= f^! g^t \mu && \in \widetilde{K}^0(D(2t,n)),\\
\beta &= (sf)^! g^{t+1}\mu && \in \widetilde{K}^{-1}(D(2t+1,n)),\\
g' &= (sf)^! g^{t+1} && \text{and } \nu_1 = p^! \nu \in p^! \widetilde{K}^*(RP(m)),\\
\lambda_0 &= p^! \lambda && \in p^! \widetilde{KO}^0(RP(m)) \subset \widetilde{KO}^0(D(m,n)),
\end{aligned}$$

where g is the generator of $\widetilde{K}^0(S^2)$ given by the reduced Hopf bundle, ε is the complexification and ρ is the real restriction.

By [6, Theorem (3.14)], we have

Theorem (3.1). i) *$\widetilde{K}^0(2t, 2r)$ is the free abelian group generatedb y $\alpha, \alpha^2, \cdots, \alpha^r, \gamma, \gamma\alpha, \cdots, \gamma\alpha^{r-1}$.*

ii) $\widetilde{K}^{-1}(2t, 2r) = 0$

iii) *$\widetilde{K}^0(2t+1, 2r)$ is the free abelian group generated by $\alpha, \cdots, \alpha^r$.*

iv) *$\widetilde{K}^{-1}(2t+1, 2r)$ is the free abelian group generated by $\beta, \beta\alpha, \cdots, \beta\alpha^{r-1}$.*

Also, by [8, Theorem 2], we have

Theorem (3.2). i) $\widetilde{K}^*(CP(n)) = Z[\mu]/\mu^{n+1}$.

ii) *$\widetilde{KO}^0(S^{2i} \wedge CP(2r))$ is the free abelian group generated by $\mu_i, \mu_i\mu_0, \cdots, \mu_i\mu_0^{r-1}$.*

iii) $\widetilde{KO}^0(S^{2i-1} \wedge CP(2r)) = 0$.

The following lemmas are useful to introduce the generaters of $\widetilde{KO}^{-i}(D(m, n))$.

Lemma (3.3). *We have the following relations:*

$$(1) \qquad \bar{\gamma} = \begin{cases} -\gamma & (t: \textit{even}) \\ \gamma & (t: \textit{odd}) \end{cases} \qquad \textit{in } \widetilde{K}^0(2t, 2r),$$

$$(2) \qquad \bar{\beta} = \begin{cases} -\beta & (t: \textit{even}) \\ \beta & (t: \textit{odd}) \end{cases} \qquad \textit{in } \widetilde{K}^{-1}(2t+1, 2r),$$

where $\bar{a}$ means the conjugation of a.

Proof. By [8, Lemma (1.2)], we have

$$\bar{\gamma} = \begin{cases} f^! g^t \bar{\mu} & (t : \text{even}) \\ -f^! g^t \bar{\mu} & (t : \text{odd}), \end{cases}$$

$$\bar{\beta} = \begin{cases} -(sf)^! g^{t+1} \bar{\mu} & (t : \text{even}) \\ (sf)^! g^{t+1} \bar{\mu} & (t : \text{odd}). \end{cases}$$

Since $\tilde{K}^0(2t, 2r)$ and $\tilde{K}^{-1}(D(2t+1, 2r))$ are free, the Chern charactors

$$\text{ch} : \tilde{K}^0(2t, 2r) \longrightarrow \tilde{H}^*(D(2t, 2r)/D(2t, 0) ; Q) \subset \tilde{H}^*(D(2t, 2r) ; Q),$$

$$\text{ch} : \tilde{K}^{-1}(D(2t+1, 2r)) \longrightarrow \tilde{H}^*(S^1 \wedge D(2t+1, 2r) ; Q)$$

are monomorphic. Moreover, by [6, Corollary (1. 11)], we have

$$\begin{aligned} \text{ch}\, f^! g^t \bar{\mu} &= f^*(s_{2t} \wedge (-x + x^2/2! - \cdots + x^{2r}/(2r)!)) \\ &= -b(1 + a/3! + \cdots + a^{r-1}/(2r-1)!) \\ &= -\text{ch}\, f^! g^t \mu. \end{aligned}$$

$$\begin{aligned} \text{ch}(sf)^! g^{t+1} \bar{\mu} &= s \wedge f^*(s_{2t+1} \wedge (-x + x^2/2! - \cdots + x^{2r}/(2r)!)) \\ &= s \wedge b'(a/2! + \cdots + a^r/(2r)!) \\ &= \text{ch}(sf)^! g^{t+1} \mu. \end{aligned}$$

Therefore we have the results.

Lemma (3. 4). *For* $\gamma\alpha^{k-1} \in \tilde{K}^0(D(2t, 2r))$ *and* $\beta\alpha^{k-1} \in \tilde{K}^{-1}(D(2t+1, 2r))$ *we have the following formulas :*

(1) $\delta(\gamma\alpha^{k-1}) = g^t(\mu - \bar{\mu})(\mu + \bar{\mu})^{k-1}$

(2) $\delta(\beta\alpha^{k-1}) = g^{t+1}(\mu + \bar{\mu})^k$,

where δ *is the homomorphism in* (1. 2).

Proof. By [6, Corollary (1. 11) and Lemma (3. 6)] we have

$$\begin{aligned} \text{ch}\, \delta(\gamma\alpha^{k-1}) &= \delta 2^{k-1} b(1 + a/3! + \cdots + a^{r-1}/(2r-1)!)(a/2! + \cdots + a^r/(2r)!)^{k-1} \\ &= 2^k(s_{2t} \wedge (x + x^3/3! + \cdots + x^{2r-1}/(2r-1)!) \\ &\qquad \times (x^2/2! + \cdots + x^{2r}/(2r)!)^{k-1}) \\ &= \text{ch}\, g^t(\mu - \bar{\mu})(\mu + \mu)^{k-1}. \end{aligned}$$

Since $\tilde{K}^*(CP(2r))$ is free, ch : $\tilde{K}^*(CP(2r)) \longrightarrow \tilde{H}^*(CP(2r) ; Q)$ is monomorphic. Therefore, we have the formula (1).

Similarly to the above, we have the formula (2).

4. The rank of $\widetilde{KO}^{-i}(m, 2r)$

In this section, we determine the rank of $\widetilde{KO}^{-i}(m, 2r)$ in Proposition (4. 8). First we have the following lemmas.

Lemma (4. 1). *Every torsion element in* $\widetilde{KO}^{-i}(m, 2r)$ *is of order* 2.

Proof. Since $\widetilde{K}^{-4}(m, 2r)$ is free and $\rho\varepsilon=2$, we have the result.

Lemma (4. 2). *Let* $i: D(m, n)\subset D(m', n')$ $(m\leqq m',\ n\leqq n')$ *be the inclusion, and* z *be a generator of* $\widetilde{KO}(S^4)$. *Then we have*

i) $i^!(\alpha_0^k)=\alpha_0^k$ *and* $i^!(z\alpha_0^k)=z\alpha_0^k$.

Especially, for $m=0$, *we have*

ii) $i^!(\alpha_0^k)=\mu_0^k$ and $i^!(z\alpha_0^k)=z\mu_0^k=2\mu_2\mu_0^{k-1}$.

Proof. Since $i^!$ is a ring homomorphism and also a homomorphism of $\widetilde{KO}^*(point)$-module, i) is trivial from the construction of α_0.

If $m=0$, by [6, Theorem (2. 2)],

$$i^!(\alpha_0)=i^!(\gamma_1-\xi_1-1)=\rho(\gamma-1_C)=\mu_0.$$

Therefore $i^!(\alpha_0^k)=\mu_0^k$ and $i^!(z\alpha_0^k)=z\mu_0^k$. Furthermore, we have

$$\varepsilon(z\mu_0^k)=2g^2(\mu+\mu)^k,$$
$$\varepsilon(\mu_2\mu_0^{k-1})=g^2(\mu+\bar{\mu})^k.$$

Since $\varepsilon: \widetilde{KO}^{-4}(CP(2r))\longrightarrow\widetilde{K}^{-4}(CP(2r))$ is monomorphic, we have $z\mu_0^k=2\mu_2\mu_0^{k-1}$.

We shall consider the spectral sequence in $\widetilde{KO}$-theory for $D(m, 2r)/D(m, 0)$. Then, we have

$$E_2^{p,-p-i}=\widetilde{H}^p(D(m, 2r)/D(m, 0)\,;\ KO^{-p-i}(point)).$$

By Theorem 1 and [6, Proposition (1. 6) and Theorem (1. 9)], we can enumerate $E_2^{p,-p-i}$ for $i=0, 1, 2, \cdots, 7$; and we obtain the following results as for the rank of $\sum_p E_2^{p,-p-i}$:

(4. 3)

i \ $(m, 2r)$	$(4t,\ 2r)$	$(4t+1, 2r)$	$(4t+2, 2r)$	$(4t+3, 2r)$
0 (mod 4)	r	r	$2r$	r
1 (mod 4)	0	0	0	r
2 (mod 4)	r	0	0	0
3 (mod 4)	0	r	0	0

Then, the rank of $\widetilde{KO}^{-i}(m, 2r)$ is at most as the above.

Next, we shall show that the rank of $\widetilde{KO}^{-i}(m, 2r)$ is no less than that of $\sum_p E_2^{p,-p-i}$. The element α_0 of $\widetilde{KO}^0(D(m, n))$ belongs to the direct summand $\widetilde{KO}^0(m, n)$, because $i^!\alpha_0=0$ in the exact sequence (1. 1) (cf. [6, Theorem (2. 2)]). Therefore, by Lemma (4. 2), ii), and Theorem (3. 2), ii), $\widetilde{KO}^0(m, 2r)$ and $\widetilde{KO}^{-4}(m, 2r)$ have r independent elements $\alpha_0, \cdots, \alpha_0^r$ and

$z\alpha_0, \cdots, z\alpha_0^r$ respectively.

In case of $m=4t+2$, consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{KO}^{-4j}(4t+2,\ 2r) & \xrightarrow{\delta} & \widetilde{KO}^{-4j+1}(S^{4t+3}\wedge CP(2r)) \\ \rho\uparrow\ \downarrow\varepsilon & & \rho\uparrow\ \downarrow\varepsilon \\ \widetilde{K}^{-4j}(4t+2, 2r) & \xrightarrow{\delta} & \widetilde{K}^{-4j+1}(S^{4t+3}\wedge CP(2r)), \end{array}$$

where δ is the homomorphism in (1. 2) and $j=0$ or 1. Let $l=2j+2t+1$, then by Lemma (3. 4)

$$\delta\rho g^{2j}\gamma\alpha^{k-1}=\rho\delta g^{2j}\gamma\alpha^{k-1}=\rho g^l(\mu-\bar{\mu})(\mu+\bar{\mu})^{k-1}=2\mu_l\mu_0^{k-1}.$$

Therefore, there are r independent elements $\rho g^{2j}\gamma, \rho g^{2j}\gamma\alpha, \cdots, \rho g^{2j}\gamma\alpha^{r-1}$ in $\widetilde{KO}^{-4j}(4t+2, 2r)$. That is, $\widetilde{KO}^{-4j}(4t+2, 2r)$ has $2r$ independent elements. We put

$$\gamma_{4j,4t+2}^k = \rho g^{2j}\gamma\alpha^{k-1} \qquad (k = 1, \cdots, r).$$

Consider the following commutative diagram

$$\begin{array}{ccc} \widetilde{KO}^{-4j}(S^{4t+2}\wedge CP(2r)) & \xrightarrow{f^!} & \widetilde{KO}^{-4j}(4t+2,\ 2r) \\ \rho\uparrow\ \downarrow\varepsilon & & \rho\uparrow\ \downarrow\varepsilon \\ \widetilde{K}^{-4j}(S^{4t+2}\wedge CP(2r)) & \xrightarrow{f^!} & \widetilde{K}^{-4j}(4t+2,\ 2r), \end{array}$$

where $f^!$ is the homomorphism in (1. 2). Since

$$f^!(g^l\mu(\mu+\bar{\mu})^{k-1}) = g^{2j}\gamma_k = g^{2j}\gamma\alpha^{k-1} \qquad \text{(cf. [6, (3. 9)])},$$

we have

$$\gamma_{4j,4t+2}^k = f^!\rho(g^l\mu(\mu+\bar{\mu})^{k-1}) = f^!(\mu_l\mu_0^{k-1}).$$

In summary

$$\text{(4. 4)} \qquad \begin{cases} \delta\gamma_{4j,4t+2}^k = 2\mu_l\mu_0^{k-1} \\ \gamma_{4j,4t+2}^k = f^!(\mu_l\mu_0^{k-1}). \end{cases}$$

In the same manner as the above we can define the independent elements as follows :

In case of $m = 4t + 3$, define the elements in $\widetilde{KO}^{-4j-1}(4t+3, 2r)$ by

$$\gamma_{4j+1,4t+3}^k = \rho g^{2j}\beta\alpha^{k-1} \qquad (k=1, \cdots, r),$$

Then, we have

$$\text{(4. 5)} \qquad \begin{cases} \gamma_{4j+1,4t+3}^k = f^!(\mu_l\mu_0^{k-1}) \\ \delta\gamma_{4j+1,4t+3}^k = 2\mu_l\mu_0^{k-1}, \end{cases}$$

where $l = 2j + 2t + 2$.

In case of $m = 4t$, define the elements in $\widetilde{KO}^{-4j-2}(4t, 2r)$ by

$$\gamma_{4j+2,4t}^k = \rho g^{2j+1}\gamma\alpha^{k-1} \qquad (k=1, \cdots, r),$$

then, we have

$$(4.6)\qquad \begin{cases} \gamma^k_{4j+2,4t} = f^!(\mu_l\mu_0^{k-1}) \\ \delta\gamma^k_{4j+2,4t} = 2\mu_l\mu_0^{k-1}, \end{cases}$$

where $l=2j+2t+1$.

In case of $m=4t+1$, define the elements in $\widetilde{K}O^{-4j-3}(4t+1, 2r)$ by

$$\gamma^k_{4j+3,4t+1} = \rho g^{2j+1}\beta\gamma^{k-1} \qquad (k=1, \cdots, r),$$

then, we have

$$(4.7)\qquad \begin{cases} \gamma^k_{4j+3,4t+1} = f^!(\mu_l\mu_0^{k-1}) \\ \delta\gamma^k_{4j+3,4t+1} = 2\mu_l\mu_0^{k-1}, \end{cases}$$

where $l=2j+2t+2$.

From the above mentioned facts, we have the following results:

Proposition (4.8). *The rank of $\widetilde{K}O^{-i}(m, 2r)$ is given by the table* (4.3).

5. The Bott sequence

There is an exact sequence due to Bott, which may be written as follows:

$$(5.1)\qquad \cdots \longrightarrow K^nO(X) \xrightarrow{\varepsilon} K^n(X) \xrightarrow{\rho I^{-1}} KO^{n+2}(X) \xrightarrow{d} KO^{n+1}(X) \longrightarrow \cdots,$$

where $I: K^{n+2}(X) \longrightarrow K^n(X)$ is the Bott periodicity isomorphism and d is the multiplication by the generator w of $\widetilde{K}O(S^1)$ (cf. [2], [3]). The sequence commutes with homomorphisms induced by a mapping $f: X \longrightarrow Y$, and also the homomorphisms in (1.2). In our case, ε is immediately known by § 4. As for additive homomorphism ρI^{-1}, from the observation of § 4, we have the following

Lemma (5.2). i) *In* $\rho I^{-1}: \widetilde{K}^{-4j-2}(m, 2r) \longrightarrow \widetilde{K}O^{-4j}(m, 2r)$,

$\rho I^{-1}(g\alpha^k) = 2\alpha_0^k$ (*if* $j=0$),

$\rho I^{-1}(g^3\alpha^k) \equiv z\alpha_0^k \mod 2$ (*if* $j=1$).

ii) *In* $\rho I^{-1}: \widetilde{K}^{-4j-2}(4t+2, 2r) \longrightarrow \widetilde{K}O^{-4j}(4t+2, 2r)$,

$\rho I^{-1}(g^{2j+1}\gamma\alpha^{k-1}) = \gamma^k_{4j,4t+2}$.

iii) *In* $\rho I^{-1}: \widetilde{K}^{-4j-3}(4t+3, 2r) \longrightarrow \widetilde{K}O^{-4j-1}(4t+3, 2r)$,

$\rho I^{-1}(g^{2j+1}\beta\alpha^{k-1}) = \gamma^k_{4j+1,4t+3}$.

iv) *In* $\rho I^{-1}: \widetilde{K}^{-4j-4}(4t, 2r) \longrightarrow \widetilde{K}O^{-4j-2}(4t, 2)$,

$\rho I^{-1}(g^{2j+2}\gamma\alpha^{k-1}) = \gamma^k_{4j+2,4t}$.

v) *In* $\rho I^{-1}: \tilde{K}^{-4j-2}(4t,2r) \longrightarrow \widetilde{KO}^{-4j}(4t,2r)$,
$\rho I^{-1}(g^{2j+1}\gamma\alpha^{k-1}) \equiv 0 \mod 2$.

vi) *In* $\rho I^{-1}: \tilde{K}^{-4j-5}(4t+1,2r) \longrightarrow \widetilde{KO}^{-4j-3}(4t+1,2r)$,
$\rho I^{-1}(g^{2j+2}\beta\alpha^{k-1}) = \gamma^k_{4j+3,4t+1}$.

Proof. Since $2z\alpha_0^k = \rho\varepsilon(z\alpha_0^k) = \rho(2g^2\alpha^k) = 2\rho(g^2\alpha^k)$, we have $\rho(g^2\alpha^k) \equiv z\alpha_0^k \mod 2$. i.e. $\rho I^{-1}(g^3\alpha^k) \equiv z\alpha_0^k \mod 2$.

Since $\varepsilon\rho I^{-1}(g^{2j+1}\gamma\alpha^{k-1}) = g^{2j}(\gamma+\bar{\gamma})\alpha^{k-1} = 0$ in $\tilde{K}^{-4j}(4t,2r)$ by Lemma (3.3), $2\rho I^{-1}(g^{2j+1}\gamma\alpha^{k-1}) = \rho\varepsilon\rho\beta^{-1}(g^{2j+1}\gamma\alpha^{k-1}) = 0$. i.e. $\rho I^{-1}(g^{2j+1}\gamma\alpha^{k-1}) \equiv 0 \mod 2$.

The rest is trivial.

6. Computation of $\widetilde{KO}^{-i}(m,2r)$ for m = 0, 1, 2 and 3

Since $D(0,2r) \approx CP(2r)$ and $\widetilde{KO}^{-i}(0,2r) = \widetilde{KO}^{-i}(CP(2r))$, we determine $\widetilde{KO}^{-i}(m,2r)$ for $m = 1, 2$ and 3 by the induction on m.

6.1. Considering the following exact sequence

$$0 \longrightarrow \widetilde{KO}^{-2}(1,2r) \longrightarrow \widetilde{KO}^{-2}(0,2r),$$

rank $\widetilde{KO}^{-2}(1,2r) = 0$ implies $\widetilde{KO}^{-2}(1,2r) = 0$.

In the same way as the above, we have $\widetilde{KO}^{-6}(1,2r) = 0$.

Consider the following exact sequence

$$0 \longrightarrow \widetilde{KO}^{0}(1,2r) \xrightarrow{i!} \widetilde{KO}^{0}(0,2r) \longrightarrow \widetilde{KO}^{-7}(S^1 \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-7}(1,2r) \longrightarrow 0.$$

By (4.2), $i^!(\alpha_0^k) = \mu_0^k$, therefore $i^!$ is epimorphic. Hence we have

$$\widetilde{KO}^{0}(1,2r) = <\alpha_0, \cdots, \alpha_0^r>$$

and $$\widetilde{KO}^{-7}(1,2r) = <\gamma^1_{7,1}, \cdots, \gamma^r_{7,1}>.$$

Consider the following exact sepuence

$$0 \longrightarrow \widetilde{KO}^{-4}(1,2r) \xrightarrow{i!} \widetilde{KO}^{-4}(0,2r) \longrightarrow \widetilde{KO}^{-3}(S^1 \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-3}(1,2r) \longrightarrow 0.$$

Since *rank* $\widetilde{KO}^{-4}(1,2r) = r$ and $\widetilde{KO}^{-3}(S^1 \wedge CP(2r))$ is free, $i^!$ is isomorphic. Therefore we have $\widetilde{KO}^{-4}(1,2r) = Z^{(r)}$ and there is a basis $a_1, \cdots, a_r$ such that $i^!(a_k) = \mu_2\mu_0^{k-1}$ and $2a_k = z\alpha_0^k$ by Lemma (4.2). Furthermore, we have $\tilde{K}O^{-3}(1, 2r) = <\gamma^1_{3,1}, \cdots, \gamma^r_{3,1}>$.

Consider the Bott sequence

$$\tilde{K}^{-2}(1, 2r) \xrightarrow{\rho I^{-1}} \tilde{K}O^0(1, 2r) \xrightarrow{d} \tilde{K}O^{-1}(1, 2r) \xrightarrow{\varepsilon} \tilde{K}^{-1}(1, 2r).$$

Since *rank* $\tilde{K}O^{-1}(1, 2r)=0$ and $\tilde{K}^{-1}(1, 2r)$ is free, we have $\varepsilon=0$. Furthermore $\rho I^{-1}(g\alpha^k)=2\alpha_0^k$ by Lemma (5. 2). Therefore we have $\tilde{K}O^{-1}(1, 2r)= Z_2^{(r)}$, which is generated by $w\alpha_0, \cdots, w\alpha_0^r$.

In the same way as the above, we have $\tilde{K}O^{-5}(1, 2r)=Z_2^{(r)}$, which is generated by $wa_1, \cdots, wa_r$.

6. 2. Consider the following exact sequence

$$0 \longrightarrow \tilde{K}O^{-1}(2, 2r) \longrightarrow \tilde{K}O^{-1}(1, 2r) \xrightarrow{\delta} \tilde{K}O^0(S^2 \wedge CP(2r)) \longrightarrow \tilde{K}O^0(2, 2r) \longrightarrow \tilde{K}O^0(1, 2r) \longrightarrow 0.$$

Since $\delta=0$, we have

$$\tilde{K}O^{-1}(2, 2r) = Z_2^{(r)}$$

and $$\tilde{K}O^0(2, 2r) = <\gamma^1_{0,2}, \cdots, \gamma^r_{0,2}, \alpha_0, \cdots, \alpha_0^r>.$$

In the same way as the above, we have

$$\tilde{K}O^{-5}(2, 2r)=Z_2^{(r)},$$

and $$\tilde{K}O^{-4}(2, 2r) = <\gamma^1_{4,2}, \cdots, \gamma^r_{4,3}, b_1, \cdots, b_r>,$$

where $i^!(b_k) = a_k$ and $2b_k = z\alpha_0^k$.

Next consider the following exact sequence

$$0 \longrightarrow \tilde{K}O^{-3}(2, 2r) \longrightarrow \tilde{K}O^{-3}(1, 2r) \xrightarrow{\delta} \tilde{K}O^{-2}(S^2 \wedge CP(2r)) \longrightarrow \tilde{K}O^{-2}(2, 2r) \longrightarrow \tilde{K}O^{-2}(1, 2r) = 0.$$

Since $\delta(\gamma^k_{3,1})=2\mu_2\mu_0^{k-1}$ by (4, 7), we have $\tilde{K}O^{-2}(2, 2r)=Z_2^{(r)}$ and $\tilde{K}O^{-3}(2, 2r) =0$.

In the same way as the above, we have $\tilde{K}O^{-6}(2, 2r) = Z_2^{(r)}$ and $\tilde{K}O^{-7}(2, 2r)=0$.

6. 3. Consider the following exact sequence

$$0 \longrightarrow \tilde{K}O^0(3, 2r) \xrightarrow{i^!} \tilde{K}O^0(2. 2r) \xrightarrow{\delta} \tilde{K}O^{-7}(S^3 \wedge CP(2r))$$

$$\longrightarrow \widetilde{KO}^{-7}(3, 2r) \longrightarrow \widetilde{KO}^{-7}(2, 2r) = 0.$$

Since *rank* $\widetilde{KO}^0(3, 2r) = r$ and $i^! \alpha_0^k = \alpha_0^k$, we have $\widetilde{KO}^0(3, 2r) = <\alpha_0, \cdots, \alpha_0^r>$. Furthermore we have $\widetilde{KO}^{-7}(3, 2r) = Z_2^{(r)}$, because $\delta(\gamma_{0,2}^k) = 2\mu_5\mu_0^{k-1}$.

Consider the following exact sequence

$$0 \longrightarrow \widetilde{KO}^{-2}(3, 2r) \longrightarrow \widetilde{KO}^{-2}(2, 2r) \longrightarrow \widetilde{KO}^{-1}(S^3 \wedge CP(2r))$$
$$\xrightarrow{f!} \widetilde{KO}^{-1}(3, 2r) \longrightarrow \widetilde{KO}^{-1}(2, 2r) \longrightarrow 0.$$

$\widetilde{KO}^{-2}(3, 2r) = Z_2^{(r)}$ is trivial. Consider the Bott sequence

$$\widetilde{K}^{-2}(3, 2r) \xrightarrow{\rho I^{-1}} \widetilde{KO}^0(3, 2r) \longrightarrow \widetilde{KO}^{-1}(3, 2r).$$

Then, since $\rho I^{-1}(g\alpha^k) = 2\alpha_0^k$, it is known that $\widetilde{KO}^{-1}(3, 2r)$ contains $Z_2^{(r)}$ as a subgroup. Therefore, by Lemma (2. 1), we have $\widetilde{KO}^{-1}(3, 2r) = Z^{(r)} + Z_2^{(r)}$, whose free part is generated by $\gamma_{1,3}^1, \cdots, \gamma_{1,3}^r$.

Now, to continue the computation, we use the following proposition which is proved in the next section.

Proposition (6.1). $\widetilde{KO}^{-3}(3, 2r) = 0$.

Consider the Bott sequence

$$0 = \widetilde{KO}^{-3}(3, 2r) \longrightarrow \widetilde{KO}^{-4}(3, 2r) \xrightarrow{\varepsilon} \widetilde{K}^{-4}(3, 2r) \longrightarrow \widetilde{KO}^{-2}(3, 2r) \longrightarrow 0,$$

then we have $\widetilde{KO}^{-4}(3, 2r) = <z\alpha_0, \cdots, z\alpha_0^r>$, because $\varepsilon(z\alpha_0^k) = 2g^2\alpha^k$.

Consider the Bott sequence

$$\widetilde{K}^{-6}(3, 2r) \xrightarrow{\rho I^{-1}} \widetilde{KO}^{-4}(3, 2r) \longrightarrow \widetilde{KO}^{-5}(3, 2r) \longrightarrow \widetilde{K}^{-5}(3, 2r)$$
$$\longrightarrow \widetilde{KO}^{-3}(3, 2r) = 0.$$

Since $\widetilde{KO}^{-4}(3, 2r)$ is free, by Lemma (5. 2), i), $\rho I^{-1}: \widetilde{K}^{-6}(3, 2r) \longrightarrow \widetilde{KO}^{-4}(3, 2r)$ is isomorphic. Therefore, $\widetilde{KO}^{-5}(3, 2r)$ is a free abelian group of rank r. Now, considering the following exact sequence

$$0 \longrightarrow \widetilde{KO}^{-5}(S^3 \wedge CP(2r)) \xrightarrow{f!} \widetilde{KO}^{-5}(3, 2r) \longrightarrow \widetilde{KO}^{-5}(2, 2r) \longrightarrow 0,$$

by Lemma (2. 2) we obtain $\widetilde{KO}^{-5}(3, 2r) = < s_{5,3}^1, \cdots, s_{5,3}^r >$, where $2s_{5,3}^k = \gamma_{5,3}^k$ $(k = 1, \cdots, r)$.

Considering the following exact sequence

$$0 \longrightarrow \widetilde{KO}^{-6}(3, 2r) \longrightarrow \widetilde{KO}^{-6}(2, 2r) \longrightarrow \widetilde{KO}^{-5}(S^3 \wedge CP(2r)),$$

$\widetilde{KO}^{-6}(3, 2r) = Z_2^{(r)}$ is trivial.

Our induction has completed.

7. Proof of Proposition (6.1)

To prove Proposition (6.1), we study the spectral sequence of $\widetilde{KO}$-* theory in detail. In general, the filtration of $\widetilde{KO}^{-3}(X)$ is given as follows:

$$\widetilde{KO}^{-3}(X) = D^{0,-3} \supset D^{1,-4} \supset \cdots \supset D^{p,-p-3} \supset \cdots \supset 0,$$

where $D^{p,-p-3} =$ kernel $(i^! : \widetilde{KO}^{-3}(X) \to \widetilde{KO}^{-3}(X^{p-1}))$ (X^p denotes the p-skeleton of X). And, E_2 and E_∞-terms are given by

$$(7.1) \quad \begin{cases} E_2^{p,-p-3} = \widetilde{H}^p(X;\ KO^{-p-3}(point)) = \begin{cases} \widetilde{H}^p(X;Z) & \text{for } p \equiv 1, 5 \pmod 8 \\ \widetilde{H}^p(X;Z_2) & \text{for } p \equiv 6, 7 \pmod 8 \end{cases} \\ E_\infty^{p,-p-3} = D^{p,-p-3}/D^{p+1,-p-4}. \end{cases}$$

The dfferentials of this spectral sequence are given by

$$(7.2) \quad \begin{cases} d_2^{p,-8t} = Sq^2 : \widetilde{H}^p(X;Z) \longrightarrow \widetilde{H}^{p+2}(X;Z_2) \\ d_2^{p,-8t-1} = Sq^2 : \widetilde{H}^p(X;Z_2) \longrightarrow \widetilde{H}^{p+2}(X;Z_2) \\ d_3^{p,-8t-2} = \delta_2 \circ Sq^2 : \widetilde{H}^p(X;Z_2) \longrightarrow \widetilde{H}^{p+3}(X;Z) \end{cases}$$

where, δ_2 is the Bockstein operator associated with the exact coefficient sequence $0 \longrightarrow Z \xrightarrow{\times 2} Z \longrightarrow Z_2 \longrightarrow 0$ (cf. [8]).

In virtue of [6, Proposition (1.6) and Theorem (1.9)], we have the following results as for E_2-terms of total degree -3 of the spectral sequence for $D(3, 2r)$:

If r is even,

$E_2^{8i+5,-8i-8} = Z_2$; generator : (c^3, d^{4i+1})

$E_2^{8i+6,-8i-9} = Z_2 + Z_2$; generators : d^{4i+3}, $c^2 d^{4i+2}$

$E_2^{8i+7,-8i-10} = Z_2 + Z_2$; generators : cd^{4i+3}, $c^3 d^{4i+2}$

$E_2^{8i+9,-8i-12} = Z_2$; generator : (c^3, d^{4i+3})

other term $= 0$,

where $i = 0, 1, \cdots, [r/2] - 1$.

If $r = 2s+1$, we can find extra terms $E_2^{8s+6,-8s-9} = Z_2$ and $E_2^{8s+7,-8s-10} = Z_2$ in addition to the above, whose generators are $c^2 d^{4s+2}$ and $c^3 d^{4s+2}$ respectively.

Also, we have the following formulas as for Sq^1 and Sq^2.

$$(7.3)\qquad \begin{cases} Sq^k(c^i)=\binom{i}{k}c^{k+i}, & & & \\ Sq^1(d)=cd, & Sq^1(d^2)=0, & Sq^1(d^3)=cd^3, & Sq^1(d^{4i})=0, \\ Sq^2(d)=d^2, & Sq^2(d^2)=c^2d^2, & Sq^2(d^3)=d^4+c^2d^3, & Sq^2(d^{4i})=0. \end{cases}$$

Since $d_2(c^3, d^{4i+1})=c^3d^{4i+2}$ by (7.2) and (7.3), the differential

$$d_2 : E_2^{8i+5,-8i-8} \longrightarrow E_2^{8i+7,-8i-9}$$

is a monomorphism. Therefore $E_3^{8i+5,-8i-8}=0$.

In the chain complex

$$E_2^{8i+4,-8i-8} \xrightarrow{d_2} E_2^{8i+6,-8i-9} \xrightarrow{d_2} E_2^{8i+8,-8i-10},$$

we have $d_2(c^0, d^{4i+2}) = c^2d^{4i+2}$ and $d_2(d^{4i+3}) = d^{4i+4} + c^2d^{4i+3}$ by (7.2) and (7.3). Therefore $E_3^{8i+6,-8i-9}=0$.

In the chain complex

$$E_2^{8i+5,-8i-9} \xrightarrow{d_2} E_2^{8i+7,-8i-10} \xrightarrow{d_2} E_2^{8i+9,-8i-11} = 0,$$

we have $d_2(cd^{4i+2}) = c^3d^{4i+2}$ and $d_2(c^3d^{4i+1}) = c^3d^{4i+2}$ by (7.2) and (7.3). Therfore $E_3^{8i+7,-8i-10} = Z_2$, whose generator is cd^{4i+3}, where $i=0, 1, \cdots, [r/2]-1$. It is trivial that $E_3^{8i+10,-8i-12} = E_2^{8i+10,-8i-12} = Z_2$ and its generator is (c^2, d^{4i+4}) for $i=0, 1, \cdots, [r/2]-1$. Since $d_3(cd^{4i+3}) = (c^2, d^{4i+4})$, the differential

$$d_3 : E_3^{8i+7,-8i-10} \longrightarrow E_3^{8i+10,-8i-12}$$

is an isomorphism. Therefore $E_4^{8i+7,-8i-10}=0$.

It is easy to see $E_3^{8i+9,-8i-12}=E_2^{8i+9,-8i-12}$. In the chain comlex

$$E_2^{8i+4,-8i-9} \xrightarrow{d_2} E_2^{8i+6,-8i-10} \xrightarrow{d_2} E_2^{8i+8,-8i-11}=0,$$

we have $d_2(d^{4i+2})=c^2d^{4i+2}$ and $d_2(c^2d^{4i+1})=c^2d^{4i+2}$. Therefore $E_3^{8i+6,-8i-10}=Z_2$, whose generator is d^{4i+3}, where $i=0, 1, \cdots, [r/2]-1$. Then the differential

$$d_3 : E_3^{8i+6,-8i-10} \longrightarrow E_3^{8i+9,-8i-12}$$

is an isomorphism, because $d_3(d^{4i+3})=(c^3, d^{4i+3})$. Therefore $E_4^{8i+9,-8i-12}=0$.

Hence we have $\widetilde{KO}^{-3}(D(3, 2r))=0$.

8. Computation of $\widetilde{KO}^{-i}(m, 2r)$ for $m > 3$

Now, we prove Theorem 3 by induction on m.

8.1. Assume Theorem 3 for $m = 8t$ ($t \geq 1$), i.e. the followings: $\widetilde{KO}^0(8t, 2r) = <\alpha_0, \cdots, \alpha_0^r>$, $\widetilde{KO}^{-1}(8t, 2r) = Z_2^{(r)}$, $\widetilde{KO}^{-2}(8t, 2r) = Z_2^{(r)} + <f_1, \cdots, f_r>$, where $\varepsilon(f_k) = g\gamma\alpha^{k-1}$ and $2f_k = \gamma_{2,8t}^k$, $\widetilde{KO}^{-3}(8t, 2r) = Z_2^{(r)}$, $\widetilde{KO}^{-4}(8t, 2r) = <z\alpha_0, \cdots, z\alpha_0^r> + Z_2^{(r)}$, $\widetilde{KO}^{-5}(8t, 2r) = 0$, $\widetilde{KO}^{-6}(8t, 2r) = <\gamma_{6,8t}^1, \cdots, \gamma_{6,8t}^r>$, $\widetilde{KO}^{-7}(8t, 2r) = 0$.

Consider the following exact sequence

$$0 \longrightarrow \widetilde{KO}^0(8t+1, 2r) \xrightarrow{i^!} \widetilde{KO}^0(8t, 2r) \longrightarrow \widetilde{KO}^{-7}(S^{8t+1} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-7}(8t+1, 2r) \longrightarrow \widetilde{KO}^{-7}(8t, 2r) = 0.$$

Then, $i^!(\alpha_0^k) = \alpha_0^k$ implies $\widetilde{KO}^0(8t+1, 2r) = <\alpha_0, \cdots, \alpha_0^r>$ and $\widetilde{KO}^{-7}(8t+1, 2r) = <\gamma_{7,8t+1}^1, \cdots, \gamma_{7,8t+1}^r>$.

Consider the following exact sequence

$$0 \longrightarrow \widetilde{KO}^{-2}(8t+1, 2r) \longrightarrow \widetilde{KO}^{-2}(8t, 2r) \xrightarrow{\delta} \widetilde{KO}^{-1}(S^{8t+1} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-1}(8t+1, 2r) \longrightarrow \widetilde{KO}^{-1}(8t, 2r) \longrightarrow 0.$$

Since $\widetilde{KO}^{-1}(S^{8t+1} \wedge CP(2r))$ is free and *rank* $\widetilde{KO}^{-2}(8t+1, 2r) = 0$, we have $\widetilde{KO}^{-2}(8t+1, 2r) = Z_2^{(r)}$. From $2f_k = \gamma_{2,8t}^k$, we have $\delta(f_k) = \mu_{4t+1}\mu_0^{k-1}$ by (4.6), because $\widetilde{KO}^{-1}(S^{8t+1} \wedge CP(2r))$ is free. Therefore $\widetilde{KO}^{-1}(8t+1, 2r) = Z_2^{(r)}$.

Consider the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-4}(8t+1, 2r) \xrightarrow{i^!} \widetilde{KO}^{-4}(8t, 2r) \longrightarrow \widetilde{KO}^{-3}(S^{8t+1} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-3}(8t+1, 2r) \longrightarrow \widetilde{KO}^{-3}(8t, 2r) \longrightarrow 0.$$

Since $\widetilde{KO}^{-3}(S^{8t+1} \wedge CP(2r))$ is free and $i^!(z\alpha_0^k) = z\alpha_0^k$, $i^!$ is an isomorphism. Therefore $\widetilde{KO}^{-4}(8t+1, 2r) = Z^{(r)} + Z_2^{(r)}$, whose free part is generated by $z\alpha_0, \cdots, z\alpha_0^r$. In the Bott sequence

$$\widetilde{KO}^{-4}(8t+1, 2r) \xrightarrow{\varepsilon_4} \widetilde{K}^{-4}(8t+1, 2r) \longrightarrow \widetilde{KO}^{-2}(8t+1, 2r) \xrightarrow{d_2} \widetilde{KO}^{-3}(8t+1, 2r) \xrightarrow{\varepsilon_3} \widetilde{K}^{-3}(8t+1, 2r) \longrightarrow \widetilde{KO}^{-1}(8t+1, 2r) \xrightarrow{d_1} \widetilde{KO}^{-2}(8t+1, 2r) \xrightarrow{\varepsilon_2} \widetilde{K}^{-2}(8t+1, 2r),$$

$\varepsilon_2 = 0$ implies that d_1 is isomorphic, and $\varepsilon_4(z\alpha_0^k) = 2g^2\alpha^k$ implies $d_2 = 0$. Therefore it is known that ε_3 is an isomorphism and $\widetilde{KO}^{-3}(8t+1, 2r)$

is a free abelian group of rank r. Now, by Lemma (2. 2), we have $\widetilde{KO}^{-3}(8t+1, 2r)=\langle s^1_{3,8t+1}, \cdots, s^r_{3,8t+1}\rangle$, where $2s^k_{3,8t+1}=\gamma^k_{3,8t+1}$.

Considering the following exact sequence

$$0 \to \widetilde{KO}^{-6}(8t+1,\ 2r) \longrightarrow \widetilde{KO}^{-6}(8t,\ 2r) \xrightarrow{\delta} \widetilde{KO}^{-5}(S^{8t+1}\wedge CP(2r)) \longrightarrow \widetilde{KO}^{-5}(8t+1,\ 2r) \longrightarrow \widetilde{KO}^{-5}(8t,\ 2r),$$

rank $\widetilde{KO}^{-6}(8t+1, 2r)=0$ implies $\widetilde{KO}^{-6}(8t+1, 2r)=0$, and $\delta(\gamma^k_{6,8t})=2\mu_{4t+3}\mu_0^{k-1}$ implies $\widetilde{KO}^{-5}(8t+1, 2r)=Z_2^{(r)}$.

8.2. Considering the following exact sequence

$$0 \longrightarrow \widetilde{KO}^{-1}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-1}(8t+1, 2r) \longrightarrow \widetilde{KO}^{0}(S^{8t+2}\wedge CP(2r)) \longrightarrow \widetilde{KO}^{0}(8t+2, 2r) \longrightarrow \widetilde{KO}^{0}(8t+1, 2r) \longrightarrow 0,$$

$\widetilde{KO}^{-1}(8t+2,\ 2r)=Z_2^{(r)}$ and $\widetilde{KO}^{0}(8t+2,\ 2r)=\langle\gamma^1_{0,8t+2}, \cdots, \gamma^r_{0,8t+2}, \alpha_0, \cdots, \alpha_0^r\rangle$ are trivial.

Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-3}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-3}(8t+1,\ 2r) \xrightarrow{\delta} \widetilde{KO}^{-2}(S^{8t+2}\wedge CP(2r)) \longrightarrow \widetilde{KO}^{-2}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-2}(8t+1,\ 2r) \longrightarrow 0,$$

rank $\widetilde{KO}^{-3}(8t+2, 2r)=0$ implies $\widetilde{KO}^{-3}(8t+2, 2r)=0$, and $\delta(s^k_{3,8t+1})=\mu_{4t+2}\mu_0^{k-1}$ implies $\widetilde{KO}^{-2}(8t+2, 2r)=Z_2^{(r)}$.

Consider the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-5}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-5}(8t+1, 2r) \longrightarrow \widetilde{KO}^{-4}(S^{8t+2}\wedge CP(2r)) \longrightarrow \widetilde{KO}^{-4}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-4}(8t+1, 2r).$$

$\widetilde{KO}^{-5}(8t+2, 2r)=Z_2^{(r)}$ is trivial. By the Bott sequence

$$0=\widetilde{KO}^{-3}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-4}(8t+2, 2r) \longrightarrow \widetilde{K}^{-4}(8t+2, 2r),$$

we have $\widetilde{KO}^{-4}(8t+2, 2r)=Z^{(2r)}$, because $\widetilde{K}^{-4}(8t+2, 2r)$ is free by Theorem (3. 1) and *rank* $\widetilde{KO}^{-4}(8t+2, 2r)=2r$. Hence, by Lemma (2. 2) we have $\widetilde{KO}^{-4}(8t+2, 2r)=\langle s^1_{4,8t+2}, \cdots, s^r_{4,8t+2}, z\alpha_0, \cdots, z\alpha_0^r\rangle$, where $2s^k_{4,8t+2}=\gamma^k_{4,8t+2}$.

Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-7}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-7}(8t+1, 2r) \xrightarrow{\delta} \widetilde{KO}^{-6}(S^{8t+2}\wedge CP(2r)) \longrightarrow \widetilde{KO}^{-6}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-6}(8t+1, 2r)=0,$$

rank $\widetilde{KO}^{-7}(8t+2, 2r)=0$ implies $\widetilde{KO}^{-7}(8t+2, 2r)=0$, and $\delta(\gamma_{7,8t+1}^{k})=2\mu_{4t+4}\mu_0^{k-1}$ implies $\widetilde{KO}^{-8}(8t+2, 2r)=Z_2^{(r)}$.

8. 3. Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^0(8t+3, 2r) \xrightarrow{i^!} \widetilde{KO}^0(8t+2, 2r) \xrightarrow{\delta} \widetilde{KO}^{-7}(S^{8t+3} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-7}(8t+3, 2r) \longrightarrow \widetilde{KO}^{-7}(8t+2, 2r)=0,$$

$i^!(\alpha_0^k)=\alpha_0^k$ and *rank* $\widetilde{KO}^0(8t+3, 2r)=r$ imply $\widetilde{KO}^0(8t+3, 2r)=<\alpha_0, \cdots, \alpha_0^r>$, and $\delta(\gamma_{0,8t+2}^k)=2\mu_{4t+5}\mu_0^{k-1}$ implies $\widetilde{KO}^{-7}(8t+3, 2r)=Z_2^{(r)}$.

Consider the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-2}(8t+3, 2r) \longrightarrow \widetilde{KO}^{-2}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-1}(S^{8t+3} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-1}(8t+3, 2r) \longrightarrow \widetilde{KO}^{-1}(8t+2, 2r) \longrightarrow 0.$$

$\widetilde{KO}^{-2}(8t+3, 2r)=Z_2^{(r)}$ is trivial. In the Bott sequence

$$\widetilde{K}^{-2}(8t+3, 2r) \xrightarrow{\rho I^{-1}} \widetilde{KO}^0(8t+3, 2r) \xrightarrow{d} \widetilde{KO}^{-1}(8t+3, 2r),$$

$\rho I^{-1}(g\alpha^k)=2\alpha_0^k$ implies that $\widetilde{KO}^{-1}(8t+3, 2r)$ contains $Z_2^{(r)}$ as a subgroup. Hence, by Lemma (2. 1) we have $\widetilde{KO}^{-1}(8t+3, 2r)=Z^{(r)}+Z_2^{(r)}$, whose free part is generated by $\gamma_{1,8t+3}^1, \cdots, \gamma_{1,8t+3}^r$.

Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-4}(8t+3, 2r) \xrightarrow{i^!} \widetilde{KO}^{-4}(8t+2, 2r) \xrightarrow{\delta} \widetilde{KO}^{-3}(S^{8t+3} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-3}(8t+3, 2r) \longrightarrow \widetilde{KO}^{-3}(8t+2, 2r)=0,$$

$i^!(z\alpha_0^k)=z\alpha_0^k$ and *rank* $\widetilde{KO}^{-4}(8t+3, 2r)=r$ imply $\widetilde{KO}^{-4}(8t+3, 2r)=<z\alpha_0, \cdots, z\alpha_0^r>$, and $\delta(s_{4,8t+2}^k)=\mu_{4t+3}\mu_0^{k-1}$ implies $\widetilde{KO}^{-3}(8t+3, 2r)=0$.

Consider the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-6}(8t+3, 2r) \longrightarrow \widetilde{KO}^{-6}(8t+2, 2r) \longrightarrow \widetilde{KO}^{-5}(S^{8t+3} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-5}(8t+3, 2r) \longrightarrow \widetilde{KO}^{-5}(8t+2, 2r) \longrightarrow 0.$$

$\widetilde{KO}^{-6}(8t+3, 2r)=Z_2^{(r)}$ is trivial. In the Bott sequence

$$\widetilde{K}^{-6}(8t+3, 2r) \xrightarrow{\rho I^{-1}} \widetilde{KO}^{-4}(8t+3, 2r) \longrightarrow \widetilde{KO}^{-5}(8t+3, 2r) \longrightarrow \widetilde{K}^{-5}(8t+3, 2r) \longrightarrow \widetilde{KO}^{-3}(8t+3, 2r)=0,$$

$\rho I^{-1}(g^3\alpha^k)=z\alpha_0^k$ implies that $\widetilde{KO}^{-5}(8t+3, 2r)$ is a free abelian group of rank r. Hence, by Lemma (2.2), we have $\widetilde{KO}^{-5}(8t+3, 2r)=\langle s^1_{5,8t+3}, \cdots, s^r_{5,8t+3}\rangle$, where $2s^k_{5,8t+3}=\gamma^k_{5,8t+3}$.

8.4. Consider the exact sequence

$$0\longrightarrow\widetilde{KO}^{-1}(8t+4, 2r)\longrightarrow\widetilde{KO}^{-1}(8t+3, 2r)\xrightarrow{\delta}\widetilde{KO}^0(S^{8t+4}\wedge CP(2r))$$
$$\longrightarrow\widetilde{KO}^0(8t+4, 2r)\xrightarrow{i^!}\widetilde{KO}^0(8t+3, 2r)\longrightarrow 0.$$

Since $\widetilde{KO}^0(S^{8t+4}\wedge CP(2r))$ is free and *rank* $\widetilde{KO}^{-1}(8t+4, 2r)=0$, we have $\widetilde{KO}^{-1}(8t+4, 2r)=Z_2^{(r)}$. Furthermore, $\delta(\gamma^k_{1,8t+3})=2\mu_{4t+2}\mu_0^{k-1}$ and $i^!(\alpha_0^k)=\alpha_0^k$ imply $\widetilde{KO}^0(8t+4, 2r)=Z_2^{(r)}+Z^{(r)}$, whose free part is generated by $\alpha_0, \cdots, \alpha_0^r$.

Consider the exact sequence

$$0\longrightarrow\widetilde{KO}^{-3}(8t+4, 2r)\longrightarrow\widetilde{KO}^{-3}(8t+3, 2r)\longrightarrow\widetilde{KO}^{-2}(S^{8t+4}\wedge CP(2r))$$
$$\longrightarrow\widetilde{KO}^{-2}(8t+4, 2r)\longrightarrow\widetilde{KO}^{-2}(8t+3, 2r)\longrightarrow 0.$$

$\widetilde{KO}^{-3}(8t+4, 2r)=0$ is trivial. By the Bott sequence

$$0\longrightarrow\widetilde{KO}^{-1}(8t+4, 2r)\longrightarrow\widetilde{KO}^{-2}(8t+4, 2r),$$

it is known that $\widetilde{KO}^{-2}(8t+4, 2r)$ contains $Z_2^{(r)}$ as a subgroup. Hence, by Lemma (2.1) we have $\widetilde{KO}^{-2}(8t+4, 2r)=Z^{(r)}+Z_2^{(r)}$, whose free part is generated by $\gamma^1_{2,8t+4}, \cdots, \gamma^r_{2,8t+4}$.

Considering the exact sequence

$$0\longrightarrow\widetilde{KO}^{-5}(8t+4, 2r)\longrightarrow\widetilde{KO}^{-5}(8t+3, 2r)\xrightarrow{\delta}\widetilde{KO}^{-4}(S^{8t+4}\wedge CP(2r))$$
$$\longrightarrow\widetilde{KO}^{-4}(8t+4, 2r)\longrightarrow\widetilde{KO}^{-4}(8t+3, 2r)\longrightarrow 0,$$

rank $\widetilde{KO}^{-5}(8t+4, 2r)=0$ implies $\widetilde{KO}^{-5}(8t+4, 2r)=0$, and $\delta(s^k_{5,8t+8})=\mu_{4t+4}\mu_0^{k-1}$ implies $\widetilde{KO}^{-4}(8t+4, 2r)=\langle z\alpha_0, \cdots, z\alpha_0^r\rangle$.

Consider the exact sequence

$$0\longrightarrow\widetilde{KO}^{-7}(8t+4, 2r)\longrightarrow\widetilde{KO}^{-7}(8t+3, 2r)\longrightarrow\widetilde{KO}^{-6}(S^{8t+4}\wedge CP(2r))$$
$$\longrightarrow\widetilde{KO}^{-6}(8t+4, 2r)\longrightarrow\widetilde{KO}^{-6}(8t+3, 2r)\longrightarrow 0.$$

$\widetilde{KO}^{-7}(8t+4, 2r)=Z_2^{(r)}$ is trivial. By the Bott sequence

$$0=\widetilde{KO}^{-5}(8t+4, 2r)\longrightarrow\widetilde{KO}^{-6}(8t+4, 2r)\longrightarrow\widetilde{K}^{-6}(8t+4, 2r),$$

it is known that $\widetilde{KO}^{-6}(8t+4, 2r)$ is a free abelian group of rank r. Hence, by Lemma (2. 2), we have $\widetilde{KO}^{-6}(8t+4, 2r) = \langle s^1_{6,8t+4}, \cdots, s^r_{6,8t+4} \rangle$, where $2s^k_{6,8t+4} = \gamma^k_{6,8t+4}$.

8. 5. Consider the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-2}(8t+5, 2r) \longrightarrow \widetilde{KO}^{-2}(8t+4, 2r) \xrightarrow{\delta} \widetilde{KO}^{-1}(S^{8t+5} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-1}(8t+5, 2r) \longrightarrow \widetilde{KO}^{-1}(8t+4, 2r) \longrightarrow 0.$$

Since $\widetilde{KO}^{-1}(S^{8t+5} \wedge CP(2r))$ is free and *rank* $\widetilde{KO}^{-2}(8t+5, 2r) = 0$, we have $\widetilde{KO}^{-2}(8t+5, 2r) = Z_2^{(r)}$. Furthermore, $\delta(\gamma^k_{2,8t+4}) = 2\mu_{4t+3}\mu_0^{k-1}$ implies $\widetilde{KO}^{-1}(8t+5, 2r) = Z_2^{(2r)}$ by Lemma (4. 1).

Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-4}(8t+5, 2r) \longrightarrow \widetilde{KO}^{-4}(8t+4, 2r) \longrightarrow \widetilde{KO}^{-3}(S^{8t+5} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-3}(8t+5, 2r) \longrightarrow \widetilde{KO}^{-3}(8t+4, 2r) = 0,$$

$\widetilde{KO}^{-4}(8t+5, 2r) = \langle z\alpha_0, \cdots, z\alpha_0^r \rangle$ and $\widetilde{KO}^{-3}(8t+5, 2r) = \langle \gamma^1_{3,8t+5}, \cdots, \gamma^r_{3,8t+5} \rangle$ are trivial.

Considering the exact sequence

$$0 \longrightarrow \widetilde{K}^{-6}(8t+5, 2r) \longrightarrow \widetilde{KO}^{-6}(8t+4, 2r) \xrightarrow{\delta} \widetilde{KO}^{-5}(S^{8t+5} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-5}(8t+5, 2r) \longrightarrow \widetilde{KO}^{-5}(8t+4, 2r) = 0,$$

rank $\widetilde{KO}^{-6}(8t+5, 2r) = 0$ implies $\widetilde{KO}^{-6}(8t+5, 2r) = 0$, and $\delta(s^k_{6,8t+5}) = \mu_{4t+5}\mu_0^{k-1}$ implies $\widetilde{KO}^{-5}(8t+5, 2r) = 0$.

Consider the exact sequence

$$0 \longrightarrow \widetilde{KO}^0(8t+5, 2r) \xrightarrow{i^!} \widetilde{KO}^0(8t+4, 2r) \longrightarrow \widetilde{KO}^{-7}(S^{8t+5} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-7}(8t+5, 2r) \longrightarrow \widetilde{KO}^{-7}(8t+4, 2r) \longrightarrow 0.$$

Since $\widetilde{KO}^{-7}(S^{8t+5} \wedge CP(2r))$ is free and $i^!(\alpha_0^k) = \alpha_0^k$, $i^!$ is an isomorphism. Hence, $\widetilde{KO}^0(8t+5, 2r) = Z^{(r)} + Z_2^{(r)}$, whose free part is generated by $\alpha_0, \cdots, \alpha_0^r$. By the Bott sequence

$$0 \longrightarrow \widetilde{KO}^{-7}(8t+5, 2r) \longrightarrow \widetilde{K}^{-7}(8t+5, 2r) \longrightarrow \widetilde{KO}^{-5}(8t+5, 2r) = 0,$$

$\widetilde{KO}^{-7}(8t+5, 2r)$ is a free abelian group of rank r. Therefore, by Lemma (2. 2) we have $\widetilde{KO}^{-7}(8t+5, 2r) = \langle s^1_{7,8t+5}, \cdots, s^r_{7,8t+5} \rangle$, where $2s^k_{7,8t+5} = \gamma^k_{7,8t+5}$.

8. 6. Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-3}(8t+6, 2r) \longrightarrow \widetilde{KO}^{-3}(8t+5, 2r) \xrightarrow{\delta} \widetilde{KO}^{-2}(S^{8t+6} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-2}(8t+6, 2r) \longrightarrow \widetilde{KO}^{-2}(8t+5, 2r) \longrightarrow 0,$$

rank $\widetilde{KO}^{-3}(8t+6, 2r)=0$ implies $\widetilde{KO}^{-3}(8t+6, 2r)=0$, and $\delta(\gamma^k_{3,8t+5}) = 2\mu_{4t+4}\mu_0^{k-1}$ implies $\widetilde{KO}^{-2}(8t+6, 2r)=Z_2^{(2r)}$ by Lemma (4. 1).

Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-5}(8t+6, 2r) \longrightarrow \widetilde{KO}^{-5}(8t+5, 2r) \longrightarrow \widetilde{KO}^{-4}(S^{8t+6} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-4}(8t+6, 2r) \longrightarrow \widetilde{KO}^{-4}(8t+5, 2r) \longrightarrow 0,$$

$\widetilde{KO}^{-5}(8t+6, 2r)=0$ and $\widetilde{KO}^{-4}(8t+6, 2r)= <\gamma^1_{4,8t+6}, \cdots, \gamma^r_{4,8t+6}, z\alpha_0 \cdots, z\alpha_0^r>$ are trivial.

Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-7}(8t+6, 2r) \longrightarrow \widetilde{KO}^{-7}(8t+5, 2r) \xrightarrow{\delta} \widetilde{KO}^{-6}(S^{8t+6} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-6}(8t+6, 2r) \longrightarrow \widetilde{KO}^{-6}(8t+5, 2r)=0,$$

rank $\widetilde{KO}^{-7}(8t+6, 2r)=0$ implies $\widetilde{KO}^{-7}(8t+6, 2r)=0$, and $\delta(s^k_{7,8t+5}) = \mu_{4t+6}\mu_0^{k-1}$ implies $\widetilde{KO}^{-6}(8t+6, 2r)=0$.

Consider the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-1}(8t+6, 2r) \longrightarrow \widetilde{KO}^{-1}(8t+5, 2r) \longrightarrow \widetilde{KO}^{0}(S^{8t+6} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{0}(8t+6, 2r) \longrightarrow \widetilde{KO}^{0}(8t+5, 2r) \longrightarrow 0.$$

$\widetilde{KO}^{-1}(8t+6, 2r)=Z_2^{(2r)}$ is trivial. By the Bott sequence

$$0 \longrightarrow \widetilde{KO}^{0}(8t+6, 2r) \longrightarrow \widetilde{K}^{0}(8t+6, 2r) \longrightarrow \widetilde{KO}^{-6}(8t+6, 2r)=0,$$

$\widetilde{KO}^{0}(8t+6, 2r)$ is an abelian group of rank $2r$. Therefore, by Lemma (2. 2) we have $\widetilde{KO}^{0}(8t+6, 2r)= <\alpha_0, \cdots, \alpha_0^r, s^1_{0,8t+6}, \cdots, s^r_{0,8t+6}>$, where $2s^k_{0,8t+6}= \gamma^k_{0,8t+6}$.

8. 7. Consider the exact sequence

$$0 \longrightarrow \widetilde{KO}^{0}(8t+7, 2r) \xrightarrow{i^!} \widetilde{KO}^{0}(8t+6, 2r) \xrightarrow{\delta} \widetilde{KO}^{-7}(S^{8t+7} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-7}(8t+7, 2r) \longrightarrow \widetilde{KO}^{-7}(8t+6, 2r)=0.$$

$i^!(\alpha_0^k)=\alpha_0^k$ and *rank* $\widetilde{KO}^{0}(8t+7, 2r)=r$ imply $\widetilde{KO}^{0}(8t+7, 2r)= <\alpha_0, \cdots,$

$\alpha_0^r>$, and $\delta(s_{0,8t+6}^k)=\mu_{4t+7}\mu_0^{k-1}$ implies $\widetilde{KO}^{-7}(8t+7,2r)=0$.

Considering the exact sequence

$$0\longrightarrow\widetilde{KO}^{-2}(8t+7,2r)\longrightarrow\widetilde{KO}^{-2}(8t+6,2r)\longrightarrow\widetilde{KO}^{-1}(S^{8t+7}\wedge CP(2r))$$
$$\longrightarrow\widetilde{KO}^{-1}(8t+7,2r)\longrightarrow\widetilde{KO}^{-1}(8t+6,2r)\longrightarrow 0,$$

$\widetilde{KO}^{-2}(8t+7,2r)=Z_2^{(2r)}$ is trivial. In the Bott sequence

$$\widetilde{K}^{-2}(8t+7,2r)\xrightarrow{\rho I^{-1}}\widetilde{KO}^{0}(8t+7,2r)\longrightarrow\widetilde{KO}^{-1}(8t+7,2r)$$
$$\xrightarrow{\varepsilon}\widetilde{K}^{-1}(8t+7,2r)\longrightarrow\widetilde{KO}^{-7}(8t+7,2r)=0,$$

$\rho I^{-1}(g\alpha^k)=2\alpha_0^k$ implies $\widetilde{KO}^{-1}(8t+7,2r)=Z_2^{(r)}+<e_1,\cdots,e_r>$, where $\varepsilon(e_k)=\beta\alpha^{k-1}$ and $2e_k\equiv\gamma_{1,8t+7}^k$ (mod 2). For

$$\varepsilon(\gamma_{1,8t+7}^k)=\varepsilon\rho(\beta\alpha^{k-1})=(\beta+\bar{\beta})\alpha^{k-1}=2\beta\alpha^{k-1}\text{ by Lemma (3. 3)}$$
$$=2\varepsilon(e_k).$$

Hence $\gamma_{1,8t+7}^k\equiv 2e_k$ (mod 2).

Considering the exact sequence

$$0\longrightarrow\widetilde{KO}^{-4}(8t+7,2r)\xrightarrow{i^!}\widetilde{KO}^{-4}(8t+6,2r)\xrightarrow{\delta}\widetilde{KO}^{-3}(S^{8t+7}\wedge CP(2r))$$
$$\longrightarrow\widetilde{KO}^{-3}(8t+7,2r)\longrightarrow\widetilde{KO}^{-3}(8t+6,2r)=0,$$

$i^!(z\alpha_0^k)=z\alpha_0^k$ and *rank* $\widetilde{KO}^{-4}(8t+7,2r)=r$ imply $\widetilde{KO}^{-4}(8t+7,2r)=<z\alpha_0,\cdots,z\alpha_0^r>$, and $\delta(\gamma_{4,8t+6}^k)=2\mu_{4t+5}\mu_0^{k-1}$ implies $\widetilde{KO}^{-3}(8t+7,2r)=Z_2^{(r)}$.

Considering the exact sequence

$$0\longrightarrow\widetilde{KO}^{-6}(8t+7,2r)\longrightarrow\widetilde{KO}^{-6}(8t+6,2r)\longrightarrow\widetilde{KO}^{-5}(S^{8t+7}\wedge CP(2r))$$
$$\longrightarrow\widetilde{KO}^{-5}(8t+7,2r)\longrightarrow\widetilde{KO}^{-5}(8t+6,2r)=0,$$

$\widetilde{KO}^{-6}(8t+7,2r)=0$ and $\widetilde{KO}^{-5}(8t+7,2r)=<\gamma_{5,8t+7}^1,\cdots,\gamma_{5,8t+7}^r>$ are trivial.

8. 8. Considering the exact sequence

$$0\longrightarrow\widetilde{KO}^{-1}(8t+8,2r)\longrightarrow\widetilde{KO}^{-1}(8t+7,2r)\longrightarrow\widetilde{KO}^{0}(S^{8t+8}\wedge CP(2r))$$
$$\longrightarrow\widetilde{KO}^{0}(8t+8,2r)\longrightarrow\widetilde{KO}^{0}(8t+7,2r)\longrightarrow 0,$$

rank $\widetilde{KO}^{-1}(8t+8,2r)=0$ implies $\widetilde{KO}^{-1}(8t+8,2r)=Z_2^{(r)}$, and $2e_k\equiv\gamma_{1,8t+7}^k$ (mod 2) implies $\widetilde{KO}^{0}(8t+8,2r)=<\alpha_0,\cdots,\alpha_0^r>$.

Consider the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-3}(8t+8, 2r) \longrightarrow \widetilde{KO}^{-3}(8t+7, 2r) \longrightarrow \widetilde{KO}^{-2}(S^{8t+8} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-2}(8t+8, 2r) \longrightarrow \widetilde{KO}^{-2}(8t+7, 2r) \longrightarrow 0.$$

$\widetilde{KO}^{-3}(8t+8, 2r) = Z_2^{(r)}$ is trivial. In the Bott sequence

$$0 \longrightarrow \widetilde{KO}^{-1}(8t+8, 2r) \longrightarrow \widetilde{KO}^{-2}(8t+8, 2r) \xrightarrow{\varepsilon} \widetilde{K}^{-2}(8t+8, 2r) \xrightarrow{\rho I^{-1}} \widetilde{KO}^{0}(8t+8, 2r),$$

$\rho I^{-1}(g\alpha^k) = 2\alpha_0^k$ and $\rho I^{-1}(g\gamma\alpha^{k-1}) = 0$ imply $\widetilde{KO}^{-2}(8t+8, 2r) = Z_2^{(r)} + \langle f_1, \cdots, f_r \rangle$, where $\varepsilon(f_k) = g\gamma\alpha^{k-1}$ and $2f_k = \gamma_{2,8t+8}^k$ (mod 2). For

$$\varepsilon(\gamma_{2,8t+8}^k) = \varepsilon\rho(g\gamma\alpha^{k-1}) = g(\gamma - \bar{\gamma})\alpha^{k-1} = 2g\gamma\alpha^{k-1} \text{ by Lemma (3. 3)}$$
$$= 2\varepsilon(f_k)$$

Hence $\gamma_{1,8t+8}^k \equiv 2f_k$ (mod 2).

Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-5}(8t+8, 2r) \longrightarrow \widetilde{KO}^{-5}(8t+7, 2r) \longrightarrow \widetilde{KO}^{-4}(S^{8t+8} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-4}(8t+8, 2r) \longrightarrow \widetilde{KO}^{-4}(8t+7, 2r) \longrightarrow 0,$$

rank $\widetilde{KO}^{-5}(8t+8, 2r) = 0$ implies $\widetilde{KO}^{-5}(8t+8, 2r) = 0$, and $\delta(\gamma_{6,8t+7}^k) = 2\mu_{4t+6}\mu_0^{k-1}$ implies $\widetilde{KO}^{-4}(8t+8, 2r) = Z_2^{(r)} + \langle z\alpha_0, \cdots, z\alpha_0^r \rangle$.

Considering the exact sequence

$$0 \longrightarrow \widetilde{KO}^{-7}(8t+8, 2r) \longrightarrow \widetilde{KO}^{-7}(8t+7, 2r) \longrightarrow \widetilde{KO}^{-6}(S^{8t+8} \wedge CP(2r)) \longrightarrow \widetilde{KO}^{-6}(8t+8, 2r) \longrightarrow \widetilde{KO}^{-6}(8t+7, 2r) = 0,$$

$\widetilde{KO}^{-7}(8t+8, 2r) = 0$ and $\widetilde{KO}^{-6}(8t+8, 2r) = \langle \gamma_{6,8t+8}^1, \cdots, \gamma_{6,8t+8}^r \rangle$ are trivial.

Our induction has completed.

9. Change of some generators

Now, we should like to change some generators.

In case of $m=1$, $\varepsilon: \widetilde{KO}^{-j}(1, 2r) \longrightarrow \widetilde{K}^{-j}(1, 2r)$ is monomorphic for $j=3, 4$ or 7, because $\widetilde{KO}^{-j}(1, 2r)$ is free. Furthermore, we have

$$\varepsilon(\gamma_{j,1}^k) = \varepsilon(\gamma_{j,1}^1\alpha_0^{k-1}) \qquad (j=3 \text{ or } 7) \text{ and } \varepsilon(a_k) = \varepsilon(a_1\alpha_0^{k-1}).$$

Hence

$$\gamma_{j,1}^k=\gamma_{j,1}^1\alpha_0^{k-1} \text{ and } a_k=a_1\alpha_0^{k-1} \text{ for } k=1,\cdots,r.$$

Define $\gamma_j=\gamma_{j,1}^1$ and $a=a_1$, then the Bott sequence implies the results in Theorem 3.

In case of $m=2$, $\varepsilon: \widetilde{KO}^{-4j}(2,2r)\longrightarrow\widetilde{K}^{-4j}(2,2r)$ is monomorphic for $j=0$ or 1, because $\widetilde{KO}^{-4j}(2,2r)$ is free. Furthermore, we have

$$\varepsilon(\gamma_{4j,2}^k)=\varepsilon(\gamma_{4j,2}^1\alpha_0^{k-1}) \qquad (j=0 \text{ or } 1) \text{ and } \varepsilon(b_k)=\varepsilon(b_1\alpha_0^{k-1}).$$

Hence

$$\gamma_{4j,2}^1=\gamma_{4j,2}^1\alpha_0^{k-1} \quad \text{and} \quad b_k=b_1\alpha_0^{k-1} \quad \text{for} \quad k=1,\cdots,r.$$

Define $\gamma_{4j}=\gamma_{4j,2}^1$ and $b=b_1$. Considering the Bott sequences

$$\widetilde{K}^{-4j-2}(2,2r)\longrightarrow\widetilde{KO}^{-4j}(2,2r)\longrightarrow\widetilde{KO}^{-4j-1}(2,2r)\longrightarrow 0$$

and $$0\longrightarrow\widetilde{KO}^{-4j-1}(2,2r)\longrightarrow\widetilde{KO}^{-4j-2}(2,2r),$$

we have the results in Theorem 3.

In case of $m=8t+1$, $8t+2$, $8t+5$ or $8t+6$, $\varepsilon: \widetilde{KO}^{-j}(m,2r)\longrightarrow \widetilde{K}^{-j}(m,2r)$ is monomorphic for $j\equiv m-2$ or $m-6$ (mod 8). Furthermore, we have

$$\varepsilon(s_{j,m}^k)=\varepsilon(s_{j,m}^1\alpha_0^{k-1}) \quad \text{for} \quad j\equiv m-6 \pmod 8$$

and $$\left.\begin{aligned}\varepsilon(\gamma_{j,m}^k)&=\varepsilon(\gamma_{j,m}^1\alpha_0^{k-1})\\ \varepsilon(\gamma_{j,m}^1)&=\varepsilon(zs_{j+4,m}^1)\end{aligned}\right\} \quad \text{for} \quad j\equiv m-2 \pmod 8.$$

Hence, we have

$$s_{j,m}^k=s_{j,m}^1\alpha_0^{k-1} \quad \text{for} \quad j\equiv m-6 \pmod 8$$

and $$\gamma_{j,m}^k=zs_{j-4,m}^1\alpha_0^{k-1} \quad \text{for} \quad j\equiv m-2 \pmod 8.$$

Define $s=s_{j,m}^1$ for $j=m-6$. Considering the Bott sequences, we have the results in Theorem 3.

In case of $m=8t+3$ or $8t+4$, $\varepsilon: \widetilde{KO}^{-j}(m,2r)\longrightarrow\widetilde{K}^{-j}(m,2r)$ is monomorphic for $j\equiv m-6$ (mod 8) and $\varepsilon(s_{j,m}^k)=\varepsilon(s_{j,m}^1\alpha_0^{k-1})$. Hence, we have

$$s_{j,m}^k=s_{j,m}^1\alpha_0^{k-1} \quad \text{for} \quad j\equiv m-6 \pmod 8.$$

Furthermore, in the Bott sequence

$$\widetilde{KO}^{-j+1}(m,2r)\xrightarrow{d}\widetilde{KO}^{-j}(m,2r)\xrightarrow{\varepsilon}\widetilde{K}^{-j}(m,2r),$$

we have $\widetilde{KO}^{-j}(m, 2r) = \mathrm{Im}\, d \oplus \mathrm{Im}\, \varepsilon$ and $\varepsilon(\gamma^k_{j,m}) = \varepsilon(zs^1_{j+4,m}\alpha_0^{k-1})$ for $j \equiv m-2 \pmod 8$. Therefore, we may choose $zs^1_{j+4,m}$, $zs^1_{j+4,m}\alpha_0$, $\cdots$, $zs^1_{j+4,m}\alpha_0^{r-1}$ as a free basis of $\widetilde{KO}^{-j}(m, 2r)$ for $j \equiv m-2 \pmod 8$. Define $s = s^1_{j,m}$ for $j = m-6$. Considering the Bott sequences, we have the results in Theorem 3.

In case of $m = 8t+7$ (or $8t+8$), $\varepsilon : \widetilde{KO}^{-j}(m, 2r) \longrightarrow \widetilde{K}^{-j}(m, 2r)$ is monomorphic for $j \equiv m-2 \pmod 8$ and $\varepsilon(\gamma^k_{j,m}) = \varepsilon(ze_1\alpha_0^{k-1})$ (or $\varepsilon(\gamma^k_{j,m}) = \varepsilon(zf_1\alpha_0^{k-1})$). Hence we have

$$\gamma^k_{j,m} = ze_1\alpha_0^{k-1} \qquad (\text{or } \gamma^k_{j,m} = zf_1\alpha_0^{k-1}) \qquad \text{for } j \equiv m-2 \pmod 8.$$

Furthermore, in the Bott sequence

$$\longrightarrow \widetilde{KO}^{-j+1}(m, 2r) \xrightarrow{d} \widetilde{KO}^{-j}(m, 2r) \xrightarrow{\varepsilon} \widetilde{K}^{-j}(m, 2r),$$

we have $\widetilde{KO}^{-j}(m, 2r) = \mathrm{Im}\, d \oplus \mathrm{Im}\, \varepsilon$ and $\varepsilon(e_k) = \varepsilon(e_1\alpha_0^{k-1})$ (or $\varepsilon(f_k) = \varepsilon(f_1\alpha_0^{k-1})$) for $j \equiv m-6 \pmod 8$. Therefore, we may choose e_1, $e_1\alpha_0$, $\cdots$, $e_1\alpha_0^{r-1}$ (or f_1, $f_1\alpha_0$, $\cdots$, $f_1\alpha_0^{r-1}$) as a free basis of $\widetilde{KO}^{-j}(m, 2r)$ for $j \equiv m-6 \pmod 8$. Define $s = e_1$ (or f_1). Considering the Bott sequences, we have the results.

This completes the proof of Theorem 3 and Theorem 4.

10. Proof of Theorem 2

In order to prove the theorem, we show the following

Lemma (10. 1). *We can define* $p : \widetilde{KO}^{-j}(m, 2r) \longrightarrow \widetilde{KO}^{-j}(m, 2r+2)$ *such that* $i^! \circ p = \mathit{identity}$, *where* $i : D(m, 2r) \subset D(m, 2r+2)$.

Proof. If $m \geqq 3$, define p by

$$p(\alpha_0^k) = \alpha_0^k,\quad p(s_{j,m}\alpha_0^{k-1}) = s_{j,m}\alpha_0^{k-1},\quad p(zs_{j,m}\alpha_0^{k-1}) = zs_{j,m}\alpha_0^{k-1}$$
$$p(z\alpha_0^{k-1}) = z\alpha_0^{k-1} \quad \text{for } 1 \leqq k \leqq r,$$

and

$$p(w\alpha_0^k) = w\alpha_0^k,\quad p(w^2\alpha_0^k) = w^2\alpha_0^k,\quad p(ws_{j,m}\alpha_0^{k-1}) = ws_{j,m}\alpha_0^{k-1}$$
$$p(w^2s_{j,m}\alpha_0^{k-1}) = w^2s_{j,m}\alpha_0^{k-1} \quad \text{for } 1 \leqq k \leqq r.$$

Then, $i^! \circ p = \text{identity}$.

Similarly for the cases $m = 1$ and 2.

The inclusion $i : D(m, 2r) \subset D(m, 2r+2)$ is decomposed as $i = i_2 \circ i_1$, where $i_1 : D(m, 2r) \subset D(m, 2r+1)$ and $i_2 : D(m, 2r+1) \subset D(m, 2r+2)$.

Then we have identity $= i^! \circ p = (i_1{}^! \circ i_2{}^!) \circ p = (i_1{}^!) \circ (i_2{}^! \circ p)$. Hence, $\kappa = i_2{}^! \circ p$ is the splitting homomorphism of the following exact sequence

$$\longrightarrow \widetilde{KO}^{-1}(D(m, 2r+1)/D(m, 2r)) \longrightarrow \widetilde{KO}^{-1}(m, 2r+1) \underset{\kappa}{\overset{i^!}{\rightleftarrows}} \widetilde{KO}^{-1}(m, 2r) \longrightarrow.$$

This completes the proof of Theorem 2.

11. Ring structures of $\widetilde{KO}^0(D(m, n))$

In this section we shall prove Theorem 6.

11. 1. Since $p^! : \widetilde{KO}^0(RP(m)) \longrightarrow \widetilde{KO}^0(D(m, n))$ is monomorphic (cf. § 1), the relations $\lambda_0^2 = -2\lambda_0$ and $\lambda_0^{f+1} = 0$ follow from those in $\widetilde{KO}^0(RP(m))$.

Since $\lambda_0\alpha_0 = (\xi_1 - 1)\otimes(\gamma_1 - \xi_1 - 1) = -(\xi_1 \otimes \xi_1 - 1)$ lies in $p^!\widetilde{KO}^0(RP(m))$ and $i^!\alpha_0 = 0$ (cf. [6, Theorem (2. 2)]), $\lambda_0\alpha_0 = p^! i^!(\lambda_0\alpha_0) = 0$, where i is the inclusion defined in (1. 1).

11. 2. In this section we discuss on the case of $n = 2r$. Since $\varepsilon(\alpha_0^{r+1}) = \alpha^{r+1} = 0$ and $\varepsilon(\zeta\alpha_0^r) = (1/2)(\gamma + \bar{\gamma})\alpha^r = \gamma\alpha^r = 0$ (for $m = 8t+6$) ($\varepsilon(\zeta\alpha_0^r) = (\gamma + \bar{\gamma})\alpha^r = 2\gamma\alpha^r = 0$ (for $m = 8t+2$)) (cf. [7, Theorem 3]), $2\alpha_0^{r+1} = \rho\varepsilon(\alpha_0^{r+1}) = 0$ and $2\zeta\alpha_0^r = \rho\varepsilon(\zeta\alpha_0^r) = 0$. Hence, α_0^{r+1} and $\zeta\alpha_0^r$ lie in the torsion part of $\widetilde{KO}^0(D(m, 2r))$. Therefore, in case of $m = 8t$, $8t+1$, $8t+3$ or $8t+7$ ($m = 8t+2$ or $8t+6$), α_0^{r+1} lies (α_0^{r+1} and $\zeta\alpha_0^r$ lie) in $p^!\widetilde{KO}^0(RP(m))$ and the relation $i^!\alpha_0 = 0$ implies $\alpha_0^{r+1} = p^! i^!(\alpha_0^{r+1}) = 0$ ($\alpha_0^{r+1} = 0$ and $\zeta\alpha_0^r = p^! i^!(\zeta\alpha_0^r) = 0$).

Moreover, in case of $m = 8t+6$ ($m = 8t+2$), since $2\zeta^2 = \rho\varepsilon(\zeta^2) = \rho\gamma^2 = 0$ ($2\zeta^2 = \rho(4\gamma^2) = 0$) (cf. [7, Theorem 3]), ζ^2 lies in $p^!\widetilde{KO}^0(RP(m))$. Also $\lambda_0\zeta$ lies in $p^!\widetilde{KO}^0(RP(m))$. Considering the following commutative diagram

$$\begin{array}{ccc}
\widetilde{KO}^0(S^m \wedge CP(2r)^+) & \xrightarrow{f^!} & \widetilde{KO}^0(D(m, 2r)) \\
\hat{i}\downarrow \uparrow \hat{p}^! & & i^! \downarrow \uparrow p^! \\
\widetilde{KO}^0(S^m) & \xrightarrow{f^!} & \widetilde{KO}^0(RP(m)),
\end{array}$$

we have $i^!\zeta = i^! f^! \mu_{4t+3} = f^! \hat{i}^! \mu_{4t+3} = 0$ ($i^!\zeta = i^! f^! \mu_{4t+1} = f^! \hat{i}^! \mu_{4t+1} = 0$). Therefore we have $\zeta^2 = p^! i^! \zeta^2 = 0$ and $\lambda_0\zeta = p^! i^!(\lambda_0\zeta) = 0$.

In case of $m = 8t+5$, $i^! : \widetilde{KO}^0(D(8t+6, 2r)) \longrightarrow \widetilde{KO}^0(D(8t+5, 2r))$ is epimorphic and $i^!\alpha_0^k = \alpha_0^k$, $i^!(\zeta\alpha_0^k) = \theta\alpha_0^k$. Therefore, the relations $\alpha_0^{r+1} = 0$ and $\zeta\alpha_0^r = 0$ in $\widetilde{KO}^0(D(8t+6, 2r))$ imply the relations $\alpha_0^{r+1} = 0$ and $\theta\alpha_0^r = 0$ in $\widetilde{KO}^0(D(8t+5, 2r))$. Moreover we have $\lambda_0\theta = i^!(\lambda_0\zeta) = 0$ and $\theta^2 = i^!(\zeta^2) = 0$.

In case of $m=8t+4$, $i^! : \widetilde{KO}^0(D(8t+5, 2r)) \longrightarrow \widetilde{KO}^0(D(8t+4, 2r))$ is isomorphic. Therefore, all the relations in $\widetilde{KO}^0(D(8t+4, 2r))$ follow from those in $\widetilde{KO}^0(D(8t+5, 2r))$.

11. 3. In this section we discuss on the case of $n=4r+1$. By Theorem 1 and Theorem 2, we have

$$\widetilde{KO}^0(D(m, 4r+1)) = \widetilde{KO}^0(D(m, 4r)) + \widetilde{KO}^0(D(m, 4r+1)/D(m, 4r)),$$

and by [9], the groups $\widetilde{KO}^0(D(m, 4r+1)/D(m, 4r))$ are as 3) of Theorem 5. As for the generators of the groups we have the following

Lemma (11 .3). *α_0^{2r+1} is a generator of the torsion part of the summand $\widetilde{KO}^0(D(m, 4r+1)/D(m, 4r))$.*

Proof. By Lemma (4. 2), we have $i^!(\alpha_0^k)=\mu_0^k$ by the homomorphism $i^! : \widetilde{KO}^0(D(m, 4r+1)) \longrightarrow \widetilde{KO}^0(CP(4r+1))$. Since $\mu_0^{2r+1} \neq 0$ in $\widetilde{KO}^0(CP(4r+1))$ (cf. [8]), the element α_0^{2r+1} is not zero in $\widetilde{KO}^0(D(m, 4r+1))$. Moreover, the element α_0^{2r+1} is a generator of the torsion part of order 2 of the summand $\widetilde{KO}^0(D(m, 4r+1)/D(m, 4r))$, because it does not belong to $\widetilde{KO}^0(D(m, 4r))$ and $2\alpha_0^{2r+1}=\rho\varepsilon(\alpha_0^{2r+1})=\rho\alpha^{2r+1}=0$ (cf. [6, Theorem 3]).

In case of $m=8t$, $8t+1$, $8t+3$ or $8t+7$, since $i^! : \widetilde{KO}^0(m, 4r+1) \longrightarrow \widetilde{KO}^0(CP(4r+1))$ is isomorphic, the relation $\alpha_0^{2r+2}=0$ is trivial.

In case of $m=8t+2$ or $8t+6$, considering the exact sequence

$$\widetilde{KO}^0(D(m, 4r+2)) \longrightarrow \widetilde{KO}^0(D(m, 4r+1)) \longrightarrow \widetilde{KO}^1(D(m, 4r+2)/D(m, 4r+1)),$$

it is easy to see that the element $\zeta\alpha_0^{2r}$ is a generator of the free part of the summand $\widetilde{KO}^0(D(m, 4r+1)/D(m, 4r)) = Z + Z_2$ and the all relations in $\widetilde{KO}^0(D(m, 4r+1))$ excepting $2\alpha_0^{2r+1}=0$ follow from those in $\widetilde{KO}^0(D(m, 4r+2))$, because $\widetilde{KO}^1(D(m, 4r+2)/D(m, 4r+1))=0$ by [9, Table (3)].

In case of $m=8t+5$, considering the exact sequence

$$\widetilde{KO}^0(S^{8t+6} \wedge CP(4r+1)^+) \xrightarrow{f^!} \widetilde{KO}^0(D(8t+6, 4r+1)) \xrightarrow{i^!} \widetilde{KO}^0(D(8t+5, 4r+1)) \longrightarrow 0,$$

it is easy to see that all the relations in $\widetilde{KO}^0(D(8t+5, 4r+1))$ excepting $\theta\alpha_0^{2r}=0$ follow from those in $\widetilde{KO}^0(D(8t+6, 4r+1))$. Also we have $\theta\alpha_0^{2r}=i^!(\zeta\alpha_0^{2r})=i^!f^!(\tau)=0$, because $\zeta\alpha_0^{2r}=(1/2)f^!\mu_3\mu_0^{2r}=f^!\tau$ (cf. [8, Theorem 2]).

In case of $m=8t+4$, $i^!: \widetilde{KO}^0(D(8t+5, 4r+1)) \longrightarrow \widetilde{KO}^0(D(8t+4, 4r+1))$ is isomorphic. Therefore, all the relations in $\widetilde{KO}^0(D(8t+4, 4r+1))$ follow from those in $\widetilde{KO}^0(D(8t+5, 4r+1))$.

11. 4. In this section we discuss on the case of $n=4r+3$. By Theorem 1 and Theorem 2, we have

$$\widetilde{KO}^0(D(m, 4r+3)) = \widetilde{KO}^0(D(m, 4r+2)) + \widetilde{KO}^0(D(m, 4r+3)/D(m, 4r+2)),$$

and by [9], the groups $\widetilde{KO}^0(D(m, 4r+3)/D(m, 4r+2))$ are as 3) of Theorem 5.

In case of $m=8t$, $8t+1$ or $8t+7$, since $\widetilde{KO}^0(D(m, 4r+3))$ is isomorphic to $\widetilde{KO}^0(D(m, 4r+2))$, the relations in $\widetilde{KO}^0(D(m, 4r+3))$ follow from those in $\widetilde{KO}^0(D(m, 4r+2))$.

In case of $m=8t+6$, since $\varepsilon(\zeta\alpha_0^{2r+1}) = \gamma\alpha^{2r+1} \neq 0$ in $\widetilde{K}^0(D(8t+6, 4r+3))$ (cf. [6, Theorem 3]), the element $\zeta\alpha_0^{2r+1}$ is a generator of the summand $\widetilde{KO}^0(D(8t+6, 4r+3)/D(8t+6, 4r+2))$.

In case of $m=8t+2$, considering the exact sequence

$$\longrightarrow \widetilde{KO}^0(S^m \wedge CP(4r+3)^+) \xrightarrow{f^!} \widetilde{KO}^0(D(m, 4r+3)) \longrightarrow \widetilde{KO}^0(D(m-1, 4r+3)) \longrightarrow,$$

there exist $f^!(\sigma)$ in $\widetilde{KO}^0(D(m, 4r+3))$ and $\varepsilon(f^!(\sigma)) = \gamma\alpha^{2r+1} \neq 0$ (cf. [6, Theorem 3]), where $2\sigma = \mu_{4t+1}\mu_0^{2r+1}$. Therefore $\zeta' = f^!(\sigma)$ is a generator of the summand $\widetilde{KO}^0(D(8t+2, 4r+3)/D(8t+2, 4r+2))$.

Moreover, since $\widetilde{KO}^0(m, 4r+3)$ is free for $m=8t+2$ or $8t+6$, we can obtain the relations in the same way as the case of $n=2r$.

In case of $m=8t+5$, considering the exact sequence

$$\widetilde{KO}^0(S^{m+1} \wedge CP(4r+3)^+) \longrightarrow \widetilde{KO}^0(D(m+1, 4r+3)) \longrightarrow \widetilde{KO}^0(D(m, 4r+3)) \longrightarrow 0,$$

it is easy to see that $\theta\alpha_0^{2r+1}$ is a generator of the summand $\widetilde{KO}^0(D(m, 4r+3)/D(m, 4r+2))$ and all the relations in $\widetilde{KO}^0(D(m, 4r+3))$ follow from those in $\widetilde{KO}^0(D(m+1, 4r+3))$.

In case of $m=8t+4$, considering the following commutative diagram

$$\begin{array}{ccc} 0\to \widetilde{KO}^0(D(m+1,\ 4r+3)/D(m+1,4r+2)) & \xrightarrow{\hat{i}!} & \widetilde{KO}^0(D(m,\ 4r+3)/D(m,4r+2)) \\ \downarrow \pi^! & & \downarrow \pi^! \\ 0\to \widetilde{KO}^0(D(m+1,\ 4r+3)) & \xrightarrow{i!} & \widetilde{KO}^0(D(m,4r+3)), \end{array}$$

it is easy to see that $i^! \theta\alpha_0^{2r+1}$ and one more element x of order 2 are generators of the summand $\widetilde{KO}^0(D(m,\ 4r+3)/D(m,\ 4r+2))$ and the relations in $\widetilde{KO}^0(D(m,\ 4r+3))$, excepting x^2, $\lambda_0 x$, ζx and $\alpha_0 x$, follow from those in $\widetilde{KO}^0(D(m+1,\ 4r+3))$. Consider the diagram of Lemma 2 of [7], in which the functor K is replaced by the functor KO, $x\alpha_0=0$ and $x\theta=0$ are trivial. Also we have $x\lambda_0=0$, $\theta\alpha_0^{2r+1}$, x or $x+\theta\alpha_0^{2r+1}$ and $x^2=0$, $\theta\alpha_0^{2r+1}$, x or $x+\theta\alpha_0^{2r+1}$.

In case of $m=8t+3$, considering the following commutative diagram

$$\begin{array}{ccccc} Z_2 & \longrightarrow & \widetilde{KO}^0(D(m+1,\ 4r+3)/D(m+1,4r+2)) & \xrightarrow{\hat{i}} & \\ \downarrow & & \downarrow \pi! & & \widetilde{KO}^0(D(m,\ 4r+3)/D(\mathrm{m},\ 4r+2)) \\ & & & & \downarrow \pi! \\ \widetilde{KO}^0(S^{m+1}\wedge CP(4r+3)^+) & \to & \widetilde{KO}^0(D(\mathrm{m}+1,\ 4r+3)) & \to & \widetilde{KO}^0(D(m,\ 4r+3)), \end{array}$$

it is easy to see that $y=\hat{i}^!(x)$ is a generator of the summaud $\widetilde{KO}^0(D(m,\ 4r+3)/D(m,4r+2))$ and the relations in $\widetilde{KO}^0(D(m,\ 4r+3))$, excepting y^2, follow from those in $\widetilde{KO}^0(D(m+1,\ 4r+3))$. Since the element y is the image of a generator of $\widetilde{KO}^0(S^{8t+8r+9})=Z_2$ by $\hat{f}^!$: $\widetilde{KO}^0(S^{8t+8r+9})\longrightarrow \widetilde{KO}^0(D(m,\ 4r+3)/D(m,\ 4r+2))$, we have $y^2=0$. Therefore, we have $x^2=0$ or $\theta\alpha_0^{2r+1}$.

This completes the proof of Theorem 6.

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