K_O-GROUPS OF THE STUNTED REAL PROJECTIVE SPACES

MICHIKAZU FUJII and TERUKO YASUI

1. Introduction

The purpose of this note is to calculate the $\widetilde{K}_O^i$-groups of the stunted real projective spaces. Our results are tabled as follows, where $RP(n)$ is the n-dimensional real projective space and (t) is the cyclic group of order t.

(1) $\widetilde{K}_O^{4m+i}(RP(4m+k)/RP(4m-1))$ $(k\geqq 0)$

k \ i	0	−1	−2	−3	−4	−5	−6	−7
$8r$								
$r=0$	(∞)	(2)	(2)	0	(∞)	0	0	0
$r\neq 0$	$(\infty)+(2^{4r})$	$(2)+(2)$	$(2)+(2)+(2)$	(2)	$(\infty)+(2^{4r})$	0	0	0
$8r+1$								
$r=0$	$(\infty)+(2)$	$(2)+(2)$	(2)	(∞)	(∞)	0	0	(∞)
$r\neq 0$	$(\infty)+(2^{4r+1})$	$(2)+(2)$	$(2)+(2)$	(∞)	$(\infty)+(2^{4r})$	0	0	(∞)
$8r+2$	$(\infty)+(2^{4r+2})$	$(2)+(2)$	$(2)+(2)$	0	$(\infty)+(2^{4r})$	0	(2)	(2)
$8r+3$	$(\infty)+(2^{4r+2})$	$(\infty)+(2)+(2)$	$(2)+(2)$	0	$(\infty)+(2^{4r})$	(∞)	$(2)+(2)$	$(2)+(2)$
$8r+4$	$(\infty)+(2^{4r+3})$	$(2)+(2)$	$(2)+(2)$	0	$(\infty)+(2^{4r+1})$	0	(2)	(2)
$8r+5$	$(\infty)+(2^{4r+3})$	$(2)+(2)$	$(2)+(2)$	(∞)	$(\infty)+(2^{4r+2})$	0	0	(∞)
$8r+6$	$(\infty)+(2^{4r+3})$	$(2)+(2)$	$(2)+(2)+(2)$	(2)	$(\infty)+(2^{4r+3})$	0	0	0
$8r+7$	$(\infty)+(2^{4r+3})$	$(\infty)+(2)+(2)$	$(2)+(2)+(2)+(2)$	$(2)+(2)$	$(\infty)+(2^{4r+3})$	(∞)	0	0

(2) $\widetilde{K}_O^{4m+i}(RP(4m+1+k)/RP(4m))$ $(k\geqq 0)$

k \ i	0	−1	−2	−3	−4	−5	−6	−7
$8r$								
$r=0$	(2)	(2)	0	(∞)	0	0	0	(∞)
$r\neq 0$	(2^{4r+1})	(2)	(2)	(∞)	(2^{4r})	0	0	(∞)
$8r+1$	(2^{4r+2})	(2)	(2)	0	(2^{4r})	0	(2)	(2)
$8r+2$	(2^{4r+2})	$(\infty)+(2)$	(2)	0	(2^{4r})	(∞)	$(2)+(2)$	$(2)+(2)$
$8r+3$	(2^{4r+3})	(2)	(2)	0	(2^{4r+1})	0	(2)	(2)
$8r+4$	(2^{4r+3})	(2)	(2)	(∞)	(2^{4r+2})	0	0	(∞)
$8r+5$	(2^{4r+3})	(2)	$(2)+(2)$	(2)	(2^{4r+3})	0	0	0
$8r+6$	(2^{4r+3})	$(\infty)+(2)$	$(2)+(2)+(2)$	$(2)+(2)$	(2^{4r+3})	(∞)	0	0
$8r+7$	(2^{4r+4})	(2)	$(2)+(2)$	(2)	(2^{4r+3})	0	0	0

(3) $\widetilde{KO}^{4m+i}(RP(4m+2+k)/RP(4m+1))$ $(k \geqq 0)$

$k \backslash i$	0	−1	−2	−3	−4	−5	−6	−7
$8r$								
$r=0$	(2)	0	(∞)	0	0	0	(∞)	(2)
$r\neq 0$	(2^{4r+1})	0	(∞)	0	(2^{4r})	0	(∞)+(2)	(2)
$8r+1$								
$r=0$	(2)	(∞)	(∞)	0	0	(∞)	(∞)+(2)	(2)+(2)
$r\neq 0$	(2^{4r+1})	(∞)	(∞)	0	(2^{4r})	(∞)	(∞)+(2)+(2)	(2)+(2)
$8r+2$	(2^{4r+2})	0	(∞)	0	(2^{4r+1})	0	(∞)+(2)	(2)
$8r+3$	(2^{4r+2})	0	(∞)	(∞)	(2^{4r+2})	0	(∞)	(∞)
$8r+4$	(2^{4r+2})	0	(∞)+(2)	(2)	(2^{4r+3})	0	(∞)	0
$8r+5$	(2^{4r+2})	(∞)	(∞)+(2)+(2)	(2)+(2)	(2^{4r+3})	(∞)	(∞)	0
$8r+6$	(2^{4r+3})	0	(∞)+(2)	(2)	(2^{4r+4})	0	(∞)	0
$8r+7$	(2^{4r+4})	0	(∞)	(∞)	(2^{4r+4})	0	(∞)	(∞)

(4) $\widetilde{KO}^{4m+i}(RP(4m+3+k)/RP(4m+2))$ $(k \geqq 0)$

$k \backslash i$	0	−1	−2	−3	−4	−5	−6	−7
$8r$								
$r=0$	0	(∞)	0	0	0	(∞)	(2)	(2)
$r\neq 0$	(2^{4r})	(∞)	0	0	(2^{4r})	(∞)+(2)	(2)+(2)+(2)	(2)+(2)
$8r+1$								
$r=0$	(2)	0	0	0	(2)	(2)	(4)	(2)
$r\neq 0$	(2^{4r+1})	0	0	0	(2^{4r+1})	(2)	(2)+(2)	(2)
$8r+2$	(2^{4r+1})	0	0	(∞)	(2^{4r+2})	(2)	(2)	(∞)
$8r+3$	(2^{4r+1})	0	(2)	(2)	(2^{4r+3})	(2)	(2)	0
$8r+4$	(2^{4r+1})	(∞)	(2)+(2)	(2)+(2)	(2^{4r+3})	(∞)+(2)	(2)	0
$8r+5$	(2^{4r+2})	0	(2)	(2)	(2^{4r+4})	(2)	(2)	0
$8r+6$	(2^{4r+3})	0	0	(∞)	(2^{4r+4})	(2)	(2)	(∞)
$8r+7$	(2^{4r+4})	0	0	0	(2^{4r+4})	(2)	(2)+(2)	(2)

2. Proof of the table (1)

Let ξ_k be the canonical line bundle over $RP(k)$ and θ^n be the trivial n-dimensional vector bundle over $RP(k)$. Then the Thom space $T(m\xi_k+\theta^n)$ and the n-fold suspension $S^n(RP(m+k)/RP(m-1))$ of the stunted projective space are homeomorphic (cf. [4, Chap. 15]).

According to [2, §12], there is the K_O-theory Thom isomorphism $\psi: K_O^{-i}(X) \cong \widetilde{K}_O^{-i}(T(\zeta))$ for $8n$-dimensional vector bundle ζ over X which admits a reduction to $Spin(8n)$. Moreover, it is well known that ζ has a spin-reduction if and only if its first and second Stiefel-Whitney classes vanish: $w_1(\zeta)=w_2(\zeta)=0$.

Since $w_1(l\xi_k)=w_2(l\xi_k)=0$ iff $l\equiv 0$ (mod 4), we have an isomorphism $K_O^i(RP(k))\cong\widetilde{K}_O^i(T(4m\xi_k+\theta^{4m}))$. Hence, we have the following

Proposition (2. 1).

$$\widetilde{K}_O^{i-4m}(RP(4m+k)/RP(4m-1))\cong K_O^i(RP(k)).$$

By Proposition (2. 1) and [3, Theorem 1], we obtain the table (1).

3. $\widetilde{K}_O^0(RP(n+k)/RP(n))$, $\widetilde{K}_O^{-4}(RP(n+k)/RP(n))$ $n\not\equiv 3$ (mod 4)

From [1, Theorem 7. 4], we have

$$\widetilde{K}_O^0(RP(n+k)/RP(n))=Z_{2^{\varphi(n+k,n)}}, \tag{3. 1}$$

where $\varphi(n+k,\ n)$ is the number of the integers s such that $n<s\leqq n+k$ and $s\equiv 0, 1, 2$ or 4 (mod 8).

Since $n\not\equiv 3$(mod 4), by [3, Theorem 1], we obtain the following short exact sequence

$$0\longrightarrow\widetilde{K}_O^{-4}(RP(n+k)/RP(n))\longrightarrow\widetilde{K}_O^{-4}(RP(n+k))\longrightarrow\widetilde{K}_O^{-4}(RP(n))\longrightarrow 0.$$

Hence, we have

$$\widetilde{K}_O^{-4}(RP(n+k)/RP(n))\cong Z_{2^{\psi(n+k,n)}}, \tag{3. 2}$$

where $\psi(n+k, n)$ is the number of the integers s such that $n<s\leqq n+k$ and $s\equiv 0$, 4, 5 or 6 (mod 8).

From (3. 1) and (3. 2), we obtain the parts of $i=0$ and -4 in the tables (2), (3) and (4).

4. $\widetilde{K}_O^i(RP(4m+1+k)/RP(4m))$, $\widetilde{K}_O^i(RP(4m+3+k)/RP(4m+2))$ (k : odd)

By [5, Corollary to Theorem 3. 8], we have the parts of k : odd in the tables (2) and (4) except for the groups

$$\widetilde{K}_O^{4m-2}(RP(4m+8r+6)/RP(4m)), \tag{4. 1}$$

$$\widetilde{K}_O^{4m-2}(RP(4m+8r+8)/RP(4m)), \tag{4. 2}$$

$$\widetilde{K}_O^{4m+2}(RP(4m+8r+4)/RP(4m+2)), \tag{4. 3}$$

$$\widetilde{K}_O^{4m+2}(RP(4m+8r+10)/RP(4m+2)), \tag{4. 4}$$

and it is also known that these are groups of order 4.

4.1. Let $X = RP(4m+8r+6)/RP(4m)$ and $A = RP(4m+8r+6)/RP(4m-1)$, and consider the exact sequence of the triple

$$0 = \widetilde{K}_O^{4m-3}(S^{4m}) \longrightarrow \widetilde{K}_O^{4m-2}(X) \longrightarrow \widetilde{K}_O^{4m-2}(A).$$

By Proposition (2. 1) we have

$$\widetilde{K}_O^{4m-2}(A) \cong \widetilde{K}_O^{-2}(RP(8r+6)) = Z_2 + Z_2 + Z_2.$$

Therefore, we have $\widetilde{K}_O^{4m-2}(X) = Z_2 + Z_2$.

4. 2. Similarly to the proof of 4. 1, we have $\widetilde{K}_O^{4m-2}(RP(4m+8r+8)/RP(4m)) = Z_2 + Z_2$.

4. 3. Let $X = RP(4m+8r+4)/RP(4m+2)$ and $A = RP(4m+8r+4)/RP(4m+3)$, and consider the exact sequence of the triple

$$Z_2 = \widetilde{K}_O^{4m+1}(S^{4m+3}) \longrightarrow \widetilde{K}_O^{4m+2}(A) \longrightarrow \widetilde{K}_O^{4m+2}(X).$$

If $r \neq 0$, by Proposition (2. 1) we have

$$\widetilde{K}_O^{4m+2}(A) \cong \widetilde{K}_O^{-2}(RP(8r)) = Z_2 + Z_2 + Z_2.$$

Therefore, $\widetilde{K}_O^{4m+2}(X) = Z_2 + Z_2$.

If $r=0$, we have $\widetilde{K}_O^{4m+2}(X) = Z_4$ by [5, Theorem 3. 2].

4. 4 Similarly to the proof of 4. 3 in the case of $r \neq 0$, we have $K_O^{4m+2}(RP(4m+8r+10)/RP(4m+2)) = Z_2 + Z_2$.

5. Some lemmas

Lemma (5. 1). *The following homomorphisms which are induced from the inclusion* i: $RP(n) \subset RP(n+1+k)$

$$i_1 : \widetilde{K}_O^{-1}(RP(n+1+k)) \longrightarrow \widetilde{K}_O^{-1}(RP(n)) \qquad (n \geqq 2),$$
$$i_2 : \widetilde{K}_O^{-2}(RP(n+1+k)) \longrightarrow \widetilde{K}_O^{-2}(RP(n)) \qquad (n \geqq 2)$$

are non-zero.

Proof. Consider the exact sequence of the pair

$$\widetilde{K}_O^{-1}(RP(8r+6)/RP(2)) \longrightarrow \widetilde{K}_O^{-1}(RP(8r+6)) \xrightarrow{i_1'} \widetilde{K}_O^{-1}(RP(2)).$$

By §4, $\widetilde{K}_O^{-1}(RP(8r+6)/RP(2))=0$ and by [3, Theorem 1], two groups $\widetilde{K}_O^{-1}(RP(8r+6))$ and $\widetilde{K}_O^{-1}(RP(2))$ are Z_2. Therefore, we obtain the isomorphism i_1'.

In the same way as the above, we have the isomorphism $i_2' : \widetilde{K}_O^{-2}(RP(8r+4)) \longrightarrow \widetilde{K}_O^{-2}(RP(2))$. The rest of the proof is immediate from the isomorphisms i_1' and i_2'.

Moreover, in virtue of the proof of [3, Theorem 1, i)], we have the following

Lemma (5.2). *The following exact sequence*

$$0 \longrightarrow \widetilde{K}_O^{-1}(S^{4s+3}) \longrightarrow \widetilde{K}_O^{-1}(RP(4s+3)) \xrightarrow{i_1} \widetilde{K}_O^{-1}(RP(4s+2)) \longrightarrow 0$$

splits.

6. $\widetilde{K}_O^i(RP(4m+1+k)/RP(4m))$ (k : even)

6. 1. $k \equiv 0 \pmod 8$. If $k=0$, the results are obvious, because $RP(4m+1)/RP(4m) \approx S^{4m+1}$. Therefore, let us assume k to be non-zero. Consider the exact seqence

$$\widetilde{K}_O^{-2}(RP(4m+1+k)) \xrightarrow{i_2} \widetilde{K}^{-2}(RP(4m)) \longrightarrow \widetilde{K}_O^{-1}(RP(4m+1+k)/RP(4m))$$
$$\longrightarrow K_O^{-1}(RP(4m+1+k)) \xrightarrow{i_1} \widetilde{K}_O^{-1}(RP(4m)).$$

By Lemma (5. 1), i_1 and i_2 are non-zero, and $\widetilde{K}_O^{-1}(RP(4m))=Z_2$, $\widetilde{K}_O^{-1}(RP(4m+1+k))=Z_2$, $\widetilde{K}_O^{-2}(RP(4m))=Z_2$ (m: odd) or Z_2+Z_2 (m: even) and $\widetilde{K}_O^{-2}(RP(4m+1+k))=Z_2$. Hence, we have

$$\widetilde{K}_O^{-1}(RP(4m+1+k)/RP(4m)) = \begin{cases} 0 & (m:\ \text{odd}), \\ Z_2 & (m:\ \text{even}). \end{cases}$$

By [3, Theorem 1],

$$i' : \widetilde{K}_O^{4m-3}(RP(4m+1+k)) \longrightarrow \widetilde{K}_O^{4m-3}(RP(4m))$$

is zero homomorphism. Therefore, considering the exact sequence of the pair, we can easily obtain the rest of this case.

6. 2. $k \equiv 2 \pmod 8$. Consider the exact sequence

$$\widetilde{K}_O^{-2}(RP(4m+1+k)) \xrightarrow{i_2} \widetilde{K}_O^{-2}(RP(4m)) \longrightarrow \widetilde{K}_O^{-1}(RP(4m+1+k)/RP(4m))$$

$$\longrightarrow \widetilde{K}_O^{-1}(RP(4m+1+k)) \xrightarrow{i_1} \widetilde{K}_O^{-1}(RP(4m)).$$

By Lemma (5. 1), i_1 and i_2 are non-zero homomorphisms. Moreover, $i_1(Z)=0$ for $\widetilde{K}_O^{-1}(RP(4m+1+k))=Z+Z_2$ by Lemma (5. 2). Hence, we have

$$\widetilde{K}_O^{-1}(RP(4m+1+k)/RP(4m)) = \begin{cases} Z+Z_2 & (m: \text{even}), \\ Z & (m: \text{odd}). \end{cases}$$

Considering the exact sequence of the pair, we have the rest of this case.

6. 3. $k \equiv 4 \pmod 8$. Similar to the proof of 6. 1.

6. 4. $k \equiv 6 \pmod 8$. Consider the exact sequence of the pair. In this case, the following homomorphisms

$$i: \widetilde{K}_O^{-6}(RP(4m+1+k)) \longrightarrow \widetilde{K}_O^{-6}(RP(4m)) \qquad \text{for } m: \text{odd},$$
$$i: \widetilde{K}_O^{4m-3}(RP(4m+1+k)) \longrightarrow \widetilde{K}_O^{4m-3}(RP(4m))$$

are zero by [3, Theorem 1]. And the image of the homomorphism

$$i_2: \widetilde{K}_O^{-2}(RP(4m+1+k)) \longrightarrow \widetilde{K}_O^{-2}(RP(4m)) \qquad \text{for } m: \text{even}$$

is Z_2 by Lemma (5. 1). Then we can easily obtain the results except for $\widetilde{K}_O^{4m-2}(RP(4m+1+k)/RP(4m))$ and it is also known that this is the group of order 8.

Next, consider the exact sequence

$$0=\widetilde{K}_O^{4m-3}(S^{4m}) \longrightarrow \widetilde{K}_O^{4m-2}(RP(4m+1+k)/RP(4m)) \longrightarrow$$
$$\widetilde{K}_O^{4m-2}(RP(4m+1+k)/RP(4m-1)).$$

By Proposition (2. 1), we have

$$\widetilde{K}_O^{4m-2}(RP(4m+1+k)/RP(4m-1))=K_O^{-2}(RP(k+1))=Z_2+Z_2+Z_2+Z_2.$$

Therefore, we have $\widetilde{K}_O^{4m-2}(RP(4m+1+k)/RP(4m))=Z_2+Z_2+Z_2$.

This completes the table (2).

7. $\widetilde{K}_O^i(RP(4m+3+k)/RP(4m+2))$ (k : even)

In the same way as §6, we have the rest parts of the table (4) except for $\widetilde{K}_O^{4m+2}(RP(4m+5)/RP(4m+2))=0$ or Z_2. On the other hand the consideration of the E_2-terms of the spectral sequence of K_O-theory for

$RP(4m+5)/RP(4m+2)$ leads that $\widetilde{K}_O^{4m+2}(RP(4m+5)/RP(4m+2))$ has at least two elements. Hence, we have $\widetilde{K}_O^{4m+2}(RP(4m+5)/RP(4m+2))=Z_2$.

This completes the table (4).

8. $\widetilde{K}_O^i(RP(4m+2+k)/RP(4m+1))$

8. 1. $k\equiv 0 \pmod 8$. If $k=0$, since $RP(4m+2)/RP(4m+1)\approx S^{4m+2}$, the results are obvious. Therefore, let us assume k to be non-zero.

In case of $m \neq 0$, cosidering the exact sequence of the pair, we have the results except for $\widetilde{K}_O^{4m+2}(RP(4m+2+k)/RP(4m+1))=Z+Z_2$ or Z. Consider the exact sequence of the triple $(RP(8t+4m+2), RP(4m+2), RP(4m+1))$, then we have $\widetilde{K}_O^{4m+2}(RP(4m+2+k)/RP(4m+1))=Z+Z_2$.

In case of $m=0$, considering the exact sequence of the pair, we have the results except for $\widetilde{K}_O^{-6}(RP(k+2)/RP(1))$ and $\widetilde{K}_O^{-2}(RP(k+2)/RP(1))$, and it is also known that these are Z or $Z+Z_2$. Consider the exact sequence of the triple $(RP(8t+2), RP(2), RP(1))$, then we have $\widetilde{K}_O^{-6}(RP(k+2)/RP(1))=Z+Z_2$ and $\widetilde{K}_O^{-2}(RP(k+2)/RP(1))=Z$.

8. 2. $k\equiv 1 \pmod 8$. Considering the exact sequence of the pair, we have the results except for $\widetilde{K}_O^{4m+2}(RP(4m+2+k)/RP(4m+1))$ and $\widetilde{K}_O^{-2}(RP(k+2)/RP(1))$, and it is also known that the ranks of these groups are 1. Consider the exact sequence of the triple $(RP(4m+2+k), RP(4m+2), RP(4m+1))$, then we have

$$\widetilde{K}_O^{4m+2}(RP(4m+2+k)/RP(4m+1)) = \begin{cases} Z+Z_2+Z_2 & \text{if } k\neq 1, \\ Z+Z_2 & \text{if } k=1, \end{cases}$$

and

$$\widetilde{K}_O^{-2}(RP(k+2)/RP(1))=Z.$$

8. 3. $k\equiv 3 \pmod 8$. Noticing that the following homomorphisms

$$i:\ \widetilde{K}_O^{4m-3}(RP(4m+2+k))\longrightarrow \widetilde{K}_O^{4m-3}(RP(4m+1))$$
$$i:\ \widetilde{K}_O^{4m+1}(RP(4m+2+k))\longrightarrow \widetilde{K}_O^{4m+1}(RP(4m+1))$$

are zero by [3, Theorem 1], we can easily obtain the results.

8. 4. The rest is similar to the above. This completes the table (3).

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OKAYAMA UNIVERSITY
YAMAGUCHI UNIVERSITY

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