ON PRIMARY DECOMPOSITION THEORY FOR MODULES

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Recently, in his paper [2], J. W. Fisher introduced a new technique for constructing decomposition theories for left R-modules, and in [3] he used it to give necessary and sufficient conditions for the classical Lasker-Noether primary decomposition theory to exist on a left R-module M over an arbitray commutative ring or on M which has the property that nil ideals are nilpotent in each factor ring of $R/l(M)$. It should be brought to our attention that all the results in [2] and [3] are still valid for left R-modules with operator domain.

In what follows, combining Fisher's technique with Tominaga's [6], we shall study the left s-primary decomposition theory on R-R'-modules. In § 1, several definitions and preliminary results are given, and § 2 contains uniqueness theorems and a canonical decomposition theorem. Finally, in § 3, we shall present several equivalent conditions for the left s-primary decomposition theory to exist on a R-R'-module.

Throughout the present paper, R and R' will represent arbitrary rings (not necessarily with 1), and M an arbitrary non-zero R-R'-bimodule. The term "submodule" will mean an R-R'-submodule.

1. If $\mathfrak{a}$ is an ideal of R, then the intersection of all prime ideals of R containing $\mathfrak{a}$ is called the prime radical of $\mathfrak{a}$, and denoted by rad $\mathfrak{a}$. The *left primary radical* $p(M)$ of M is defined as the prime radical of $l(M)=\{x\in R\,|\,xM=0\}$. A proper submodule N of M is called a *left primary submodule* if $l(M'')\subseteq p(M/N)$ for every non-zero submodule M'' of M/N. A left primary submodule N of M is called *left s-primary* if $p(M/N)$ is nilpotent modulo $l(M/N)$. Occasionally, we regard M itself as a left s-primary submodule. To be easily seen, for the bimodule ${}_RR_R$ the notion "left s-primary submodule" coincides with that of "s-left primary ideal" in the sense of [5].

If $p(M'')$ is nilpotent modulo $l(M'')$ for every non-zero factor submodule M'' of M then M is called a *left s-module.* Evidently, every left primary submodule of a left s-module is left s-primary, and if R is left Noetherian then M is a left s-module.

If $p(N)=p(M)$ for every non-zero submodule N of M then M is said to be *left p-stable*. A left p-stable module M is said to be *left s-p-stable* if $p(M)$ is nilpotent modulo $l(M)$.

Proposition 1. *Let N be a proper submodule of M. Then the following conditions are equivalent*:

(1) *M/N is left s-p-stable.*

(2) *N is a left s-primary submodule of M.*

(3) *For every submodule M' of M with $M' \not\subseteq N$, $M' \cap N$ is a left s-primary submodule of M'.*

Proof. (2) implies (1): Let M'' be an arbitrary non-zero submodule of M/N. Then $l(M/N) \subseteq l(M'') \subseteq p(M/N)$. It follows therefore that $p(M'')=p(M/N)$.

(1) implies (3): Let M'' be an arbitrary non-zero submodule of $M'/(M' \cap N)$. Since M/N is left s-p-stable, our assertion is evident by $M'/(M' \cap N) \simeq (M'+N)/N$ and $l(M'') \supseteq l(M/N)$.

Finally, (3) implies (2) trivially.

Remark 1. In the same way as [3; Prop. 1.1] was shown, we can prove that the left primary radical of a left s-p-stable module is a prime ideal. A submodule N of M is called *left-q-primary* if $p(M/N)$ is a prime ideal. Accordingly, every left s-primary submodule of M is left q-primary.

An ideal $\mathfrak{p}$ of R is called a *left associated ideal* of M if there exists a left p-stable submodule N such that $\mathfrak{p}=p(N)$. The set of all left associated ideals of M will be denoted by $P(M)$. ($P(0)=\{R\}$ by definition.) If there exists an ideal $\mathfrak{p}$ in R such that $\{\mathfrak{p}\}=P(M'')$ for every non-zero submodule M'' of M/N then N is called a *left P-submodule* of M. Finally, M is said to be *left p-worthy* if $P(M'')$ is finite and non-empty for every non-zero factor submodule M'' of M.

Now, let N be a submodule of M, and $\mathfrak{a}$ an ideal of R. We set $\mathfrak{a}^{-1}N=\{u \in M \mid \mathfrak{a}u \subseteq N\}$, which is evidently a submodule of M. Further, we set $\mathfrak{a}^{-k}N=(\mathfrak{a}^k)^{-1}N$ and $\mathfrak{a}^{-\infty}N=\bigcup_{k=1}^{\infty} \mathfrak{a}^{-k}N$, which is called the *limit module* of N by $\mathfrak{a}$ in M. If $\mathfrak{a}^{-\infty}N=\mathfrak{a}^{-k}N$ with some k, then we say that $\mathfrak{a}^{-\infty}N$ is *accessible*. Occasionally, we consider also $N_{\mathfrak{a}}=\bigcup \mathfrak{b}^{-1}N$, where $\mathfrak{b}$ ranges over all the ideals of R not contained in $\mathfrak{a}$. (If $\mathfrak{a}=R$ then $N_{\mathfrak{a}}=N$ by definition.)

If M satisfies one of the following equivalent conditions (1) and (2), it is called a *left Artin-Rees module*:

(1) For each submodule N of M and ideal $\mathfrak{a}$ of R, there exists a positive integer h such that $\mathfrak{a}^h M \cap N \subseteq \mathfrak{a} N$.

(2) For each submodule N of M and ideal $\mathfrak{a}$ of R, there exists a positive integer h such that $N = (N + \mathfrak{a}^h M) \cap \mathfrak{a}^{-1} N$.

One may remark here that if M is a left Artin-Rees module, then so is every factor submodule of M.

The proof of the next is quite similar to that of [3; Lemma 2. 1].

Lemma 1. *If a left Artin-Rees module M is a left s-module then every left P-submodule of M is left s-primary.*

2. A finite set $\{N_i \mid i \in I\}$ of left s-primary (resp. left q-primary) submodules of M is called a *left s-primary* (resp. *left q-primary*) *decomposition* of N in M if $N = \bigcap_{i \in I} N_i$ is an irredundant representation and $p(M/N_i) \neq p(M/N_j)$ for every $i \neq j$. If every submodule of M has a left s-primary decomposition in M, then M is said to have the *left s-primary decomposition theory*. Similarly, a finite set $\{N_i \mid i \in I\}$ of left P-submodules of M is called a *left P-decomposition* of N in M if $N = \bigcap_{i \in I} N_i$ is an irredundant representation and $P(M/N_i) \neq P(M/N_j)$ for every $i \neq j$. In case every submodule of M has a left P-decomposition in M, M is said to have the *left P-decomposition theory*.

The first uniqueness theorem is given in the following:

Theorem 1. *If $\{N_i \mid i \in I\}$ is a left s-primary decomposition of N in M then it is a left P-decomposition and $P(M/N) = \{p(M/N_i) \mid i \in I\}$.*

Proof. Since M/N_i is left p-stable by Prop. 1, N_i is a left P-submodule with $P(M/N_i) = \{p(M/N_i)\}$. Now, our assertion is evident by [2; Prop. 4. 5].

Proposition 2. *Let $N_1, \cdots, N_s$ be left q-primary submodules of M, and $N = \bigcap_{i=1}^{s} N_i$.*

(a) *If each $p(M/N_i)$ is nilpotent modulo $l(M/N_i)$ then $p(M/N)$ equals $\bigcap_{i=1}^{s} p(M/N_i)$ and is nilpotent modulo $l(M/N)$, and every minimal prime divisor of $l(M/N)$ coincides with some $p(M/N_i)$.*

(b) *If every N_i is left s-primary and $p(M/N_1) = \cdots = p(M/N_s)$ then N is left s-primary.*

(c) *If every N_i is left s-primary then N has a left s-primary decomposition.*

Proof. (c) is only a combination of (a) and (b).

(a) Let $p(M/N_i)^{n_i} \subseteq l(M/N_i)$, and $n=\sum_{i=1}^{s} n_i$. Then, we have $(\bigcap_{i=1}^{s} p(M/N_i))^n \subseteq \bigcap_{i=1}^{s} l(M/N_i)=l(M/N)$.

(b) It is enough to consider the case $s=2$. If M'/N is a non-zero submodule of M/N then one of $(M'+N_i)/N_i$, say, $(M'+N_1)/N_1$, is a non-zero homomorphic image of M'/N. Hence, $l(M'/N)\subseteq l((M'+N_1)/N_1) \subseteq p(M/N_1)=p(M/N)$, and so N is left s-primary by (a).

Proposition 3. *Let* $\{N_i \mid i=1, \cdots, s\}$ *be a left s-primary decomposition of a proper submodule* N *in* M, *and* $\mathfrak{p}_i=p(M/N_i)$ $(i=1, \cdots, s)$.

(a) *An ideal* $\mathfrak{a}$ *of* R *is non-prime to* N (*i. e.* $\mathfrak{a}^{-1}N\supset N$) *if and only if* $\mathfrak{a}$ *is contained in some* $\mathfrak{p}_i$.

(b) *A prime divisor* $\mathfrak{p}$ *of* $l(M/N)$ *occurs in* $P(M/N)$ *if and only if* $\mathfrak{p}$ *is non-prime to* $N_{\mathfrak{p}}$.

Proof. (a) Suppose $\mathfrak{a}^{-1}N\supset N$. Then, $\mathfrak{a}^{-1}N \not\subseteq N_j$ and $\mathfrak{a}(\mathfrak{a}^{-1}N)\subseteq N \subseteq N_j$ for some j. It follows therefore $\mathfrak{a}\subseteq\mathfrak{p}_j$. Conversely, suppose $\mathfrak{a}\subseteq\mathfrak{p}_1$, and choose a positive integer h such that $\mathfrak{p}_1^h\subseteq l(M/N_1)$. Since $N_1'=N_2\cap \cdots \cap N_s \not\subseteq N_1$ and $\mathfrak{p}_1^h N_1' \subseteq N$, there exists the least positive integer h' such that $\mathfrak{p}_1^{h'}N_1'\subseteq N$. Then, $\mathfrak{a}^{-1}N\supseteq \mathfrak{p}_1^{h'-1}N_1'+N\supset N$.

(b) Suppose $\mathfrak{p}_1, \cdots, \mathfrak{p}_{r-1}\subseteq\mathfrak{p}=\mathfrak{p}_r$ and $\mathfrak{p}_i\not\subseteq\mathfrak{p}$ for all $i>r$. Then, $N_{\mathfrak{p}}= N_1\cap\cdots\cap N_r$ and $\mathfrak{p}=\mathfrak{p}_r$ is non-prime to $N_{\mathfrak{p}}$ by (a). Conversely, suppose that $\mathfrak{p}$ is non-prime to $N_{\mathfrak{p}}$. By Prop. 2, $\mathfrak{p}$ contains one of $\mathfrak{p}_i$'s. Accordingly, without loss of generality, we may assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_r\subseteq\mathfrak{p}$ $(r>0)$ and $\mathfrak{p}_i\not\subseteq\mathfrak{p}$ $(r+1\leqslant i\leqslant s)$. Then, $N_{\mathfrak{p}}=N_1\cap\cdots\cap N_r$ is evidently a proper submodule of M. Hence, again by (a), ($\mathfrak{p}\subseteq\mathfrak{p}_j$, and so) $\mathfrak{p}=\mathfrak{p}_j$ for some $j\leqslant r$.

Now, let $\{N_i \mid i=1, \cdots, s\}$ be a left s-primary decomposition of a submodule N in M. A subset P of $P(M/N)=\{\mathfrak{p}_i=\mathfrak{p}(M/N_i) \mid i=1, \cdots s\}$ is called an *isolated subset* of $P(M/N)$ if every $\mathfrak{p}_i$ contained in one of the members of P is a member of P. If P is an isolated subset of $P(M/N)$ then we set $N_P=\bigcap_{\mathfrak{p}_i\in P}N_i$, which is called an *isolated component* of N. Since $N_P\subseteq N_{\mathfrak{p}_i}\subseteq N_i$ for every $\mathfrak{p}_i\in P$, it follows then $N_P=\bigcap_{\mathfrak{p}_i\in P}N_{\mathfrak{p}_i}$. Combining this with Th. 1, we readily obtain the following, which is the second uniqueness theorem:

Theorem 2. *Suppose* N *has a left s-primary decomposition. Then the set of isolated components of* N *does not depend on the choice of left* *s-primary decompositions of* N.

Theorem 3. *Let* $\{N_i \mid i \in I\}$ *be a left s-primary decomposition of* N. *If* M' *is a non-zero submodule of* M *then* $\{M' \cap N_i \mid i \in I\}$ *contains a left s-primary decomposition of* $M' \cap N$ *in* M'.

Proof. We set $N'_i = M' \cap N_i$. Then, Prop. 1 shows that every N'_i different from M' is a left s-primary submodule of M' with $p(M'/N'_i) = p(M/N_i)$. Now, our assertion is obvious by Prop. 2.

Corollary 1. *If* M *has the left s-primary decomposition theory, then so does every non-zero factor submodule of* M.

Proposition 4. *If* M *has the left s-primary decomposition theory then there holds the following*:

(a) *For every submodule* N *of* M *and every ideal* $\mathfrak{a}$ *of* R, $\mathfrak{a}^{-\infty}N$ *is accessible, and if* $N = N_0 \subset N_1 \subset \cdots \subset N_n$ *is an arbitrary chain of submodules of* M *such that each* N_i *is a limit module of the preceding one in* M *then* $n \leqslant s(N)$ *with a positive integer* $s(N)$ *depending solely on* N.

(b) *Let* M'/N *be an arbitrary non-zero factor submodule of* M. *If* $\mathfrak{p}$ *is an arbitrary minimal prime divisor of* $l(M'/N)$ *then* $\mathfrak{p}^{-1}N \cap M' \supset N$.

(c) *Let* N *be a submodule of* M, *and* $\mathfrak{p}$ *a prime divisor of* $l(M/N)$. *Then the following conditions are equivalent*:

(1) $\mathfrak{p}$ *is a minimal prime divisor of* $l(M/N)$.

(2) $N_{\mathfrak{p}}$ *is left s-primary and* $\mathfrak{p}(M/N_{\mathfrak{p}}) = \mathfrak{p}$.

(3) $N_{\mathfrak{p}} = (N + \mathfrak{p}^h M)_{\mathfrak{p}}$ *for some* h.

If $\mathfrak{p}$ *is a minimal prime divisor of* $l(M/N)$ *then* $N_{\mathfrak{p}}$ *is a minimal left s-primary submodule containg* N.

Proof. (a) Let $\{N_i \mid i = 1, \cdots, s\}$ be a left s-primary decomposition of N. Without loss of generality, we may assume that $\mathfrak{a} \not\subseteq p(M/N_i)$ for $i \leqslant k$ and $\mathfrak{a} \subseteq p(M/N_i)$ for $i > k$. There exists a positive integer h such that $\mathfrak{a}^h \subseteq l(M/N_i)$ for all $i > k$. Recalling that each $p(M/N_i)$ is a prime ideal (Remark 1), we readily obtain

$$\mathfrak{a}^{-\infty}N = \mathfrak{a}^{-h}N = N_1 \cap \cdots \cap N_k.$$

Now, our assertion is obvious by Th. 1.

(b) By the validity of Cor. 1, it suffices to prove that if $\mathfrak{p}$ is a minimal prime divisor of $l(M)$ then $\mathfrak{p}^{-1}0 \neq 0$. If $\{N_i \mid i \in I\}$ is a left s-primary decomposition of 0, then $\mathfrak{p}$ coincides with some $p(M/N_i)$ (Prop. 2), and our assertion is clear by the proof of (a).

(c) By Prop. 2, M is a left s-module.

(1) implies (2): In case $\mathfrak{p}=R$, there is nothing to prove. We may assume henceforth $\mathfrak{p}\neq R$. As an easy consequence of (a), we see that $N_{\mathfrak{p}}$ is the largest submodule among those of the form $\mathfrak{b}^{-1}N$ with some ideal $\mathfrak{b}$ not contained in $\mathfrak{p}$: $N_{\mathfrak{p}}=\mathfrak{c}^{-1}N$, $\mathfrak{c}\not\subseteq\mathfrak{p}$. In order to see that $N_{\mathfrak{p}}$ is left s-primary, it suffices to prove that $p(M/N_{\mathfrak{p}})=\mathfrak{p}$. Suppose here $p(M/N_{\mathfrak{p}})\neq\mathfrak{p}$, and choose a minimal prime divisor $\mathfrak{p}'$ of $l(M/N_{\mathfrak{p}})$ such that $\mathfrak{p}'\not\subseteq\mathfrak{p}$. Then, $\mathfrak{p}'^{-1}N_{\mathfrak{p}}\supset N_{\mathfrak{p}}$ by (b). But, this is a contradiction. Hence, $N_{\mathfrak{p}}$ is a left s-primary submodule with $p(M/N_{\mathfrak{p}})=\mathfrak{p}$. Now, let N'' be a left s-primary submodule of M such that $N\subseteq N''\subseteq N_{\mathfrak{p}}$. Since $\mathfrak{c}N_{\mathfrak{p}}\subseteq N\subseteq N''$ and $\mathfrak{c}\not\subseteq p(M/N'')$, it follows $N_{\mathfrak{p}}\subseteq N''$, i.e. $N_{\mathfrak{p}}=N''$.

(2) implies (3): There exists a positive integer h such that $\mathfrak{p}^hM\subseteq N_{\mathfrak{p}}$. Then, $N_{\mathfrak{p}}\subseteq(N+\mathfrak{p}^hM)_{\mathfrak{p}}\subseteq(N_{\mathfrak{p}})_{\mathfrak{p}}=N_{\mathfrak{p}}$, which implies $N_{\mathfrak{p}}=(N+\mathfrak{p}^hM)_{\mathfrak{p}}$.

(3) implies (1): Since $p(M/N_{\mathfrak{p}})=p(M/(N+\mathfrak{p}^hM)_{\mathfrak{p}})\supseteq\mathfrak{p}^h$, we obtain $p(M/N_{\mathfrak{p}})\supseteq\mathfrak{p}$. As is well-known, $\mathfrak{p}$ contains a minimal prime divisor $\mathfrak{p}''$ of $l(M/N)$. By (1)$\Rightarrow$(2), $N_{\mathfrak{p}''}$ is a left s-primary submodule and $p(M/N_{\mathfrak{p}''})=\mathfrak{p}''$. Combining this with $N_{\mathfrak{p}''}\supseteq N_{\mathfrak{p}}$ and $\mathfrak{p}\subseteq p(M/N_{\mathfrak{p}})$, we readily obtain $\mathfrak{p}=\mathfrak{p}''$.

Next, we shall prove the following canonical decomposition theorem, which contains [1; Th. 3. 4].

Theorem 4. *Suppose M has the left s-primary decomposition theory. Let N be a submodule of M, and $P(M/N)=\{\mathfrak{p}_i\,|\,i=1,\cdots,r\}$ where $\mathfrak{p}_1,\cdots,\mathfrak{p}_r$ $(r\leqslant s)$ are the minimal prime divisors of $l(M/N)$* (cf. Prop. 2 (a)). *Then, there exists a positiae integer h such that $\{N+\mathfrak{p}_i^hM)_{\mathfrak{p}_i}\,|\,i=1,\cdots,s\}$ is a left s-primary decomposition of N and $\{N+\mathfrak{p}_i^hM\,|\,i=1,\cdots,r\}$ is a left q-primary decomposition of N.*

Proof. Let $\{N_i\,|\,i=1,\cdots,s\}$ be a left s-primary decomposition of N and $\mathfrak{p}_i=p(M/N_i)$ (Th. 1). Choose a positive integer h such that $\mathfrak{p}_i^h\subseteq l(M/N_i)$ for all i. Then, $N\subseteq N+\mathfrak{p}_i^hM\subseteq N_i$, and so $N\subseteq(N+\mathfrak{p}_i^hM)_{\mathfrak{p}_i}\subseteq N_{i\mathfrak{p}_i}=N_i$. Evidently, $\mathfrak{p}(M/(N+\mathfrak{p}_i^hM))=\mathfrak{p}(M/(N+\mathfrak{p}_i^hM)_{\mathfrak{p}_i})=\mathfrak{p}_i$. If $\mathfrak{b}M'\subseteq(N+p_i^hM)_{\mathfrak{p}_i}$ for a submodule M' of M and an ideal $\mathfrak{b}\not\subseteq\mathfrak{p}_i$ then $M'\subseteq((N+p_i^hM)_{\mathfrak{p}_i})_{\mathfrak{p}_i}=(N+\mathfrak{p}_i^hM)_{\mathfrak{p}_i}$. Hence, Th. 1 proves that $\{(N+\mathfrak{p}_i^hM)_{\mathfrak{p}_i}\,|\,i=1,\cdots,s\}$ is a left s-primary decomposition of N. The rest of the proof will be almost evident by Prop. 2 (a).

3. In this section, we shall consider the following conditions:

(A) For every submodule N of M and every ideal $\mathfrak{a}$ of R, $\mathfrak{a}^{-\infty}N$ is accessible, and if $N=N_0\subset N_1\subset\cdots\subset N_n$ is an arbitrary chain of submodules of M such that each N_i is a limit module of the preceding one in M then $n\leqslant s(N)$ with a positive integer $s(N)$ depending solely on N.

(B) For each non-zero factor submodule M'/N of M, there exists a minimal prime divisor $\mathfrak{p}$ of $l(M'/N)$ such that $\mathfrak{p}^{-1}N\cap M'\supset N$.

(C) M is left $\mathfrak{p}$-worthy.

(D) Every left P-submodule of M is left primary.

(D′) Every left P-submodule of M is left s-primary.

(E) M is a left Artin-Rees module.

(F) M is a left s-module.

(G) M has the left s-primary decomposition theory.

Lemma 2. *If M has the left s-primay decomposition theory then M is a left Artin-Rees module.*

Proof. Prop. 1 and Th. 1 enables us to apply the argument used in the proof of [3; Th. 2.7] to see this.

Lemma 3. *Suppose the condition* (A) *is satisfied. If M' is an arbitrary non-zero submodule of M then* (A) *holds good for M'.*

Proof. We claim first that if N is a submodule of M' and $\mathfrak{a}$, $\mathfrak{b}$ are ideals of R then $\mathfrak{b}^{-1}\mathfrak{a}^{-1}N\cap M'=\mathfrak{b}^{-1}(\mathfrak{a}^{-1}N\cap M')\cap M'$. This enables us to see that every limit module in M' is accessible. Now, let $N=N_0'\subset N_1'\subset\cdots\subset N_n'$ be a chain of submodules of M' such that each N_i' is the limit module of N_{i-1}' by $\mathfrak{a}_i$ in M'. If we set $N_i=\mathfrak{a}_i^{-\infty}N_{i-1}=\mathfrak{a}_i^{-k_i}N_{i-1}$ then, again by the above remark, we can easily see that $N_i'=N_i\cap M'$, which implies $n\leqslant s(N)$.

Proposition 5. (A) *together with* (B) *implies* (C) *and* (F).

Proof. (F): By the validity of Lemma 3, it suffices to prove that if N is a proper submodule of M then $p(M/N)$ is nilpotent modulo $l(M/N)$. By (B), there exists a minimal prime divisor $\mathfrak{p}$ of $l(M/N)$ such that $N\subset\mathfrak{p}^{-1}N\subseteq p(M/N)^{-1}N$. If $p(M/N)^{-1}N\neq M$, then by the same reason we have $p(M/N)^{-1}N\subset p(M/p(M/N)^{-1}N)^{-1}(p(M/N)^{-1}N)\subseteq p(M/N)^{-2}N$. Continuing the same argument, we obtain $p(M/N)^{-k}N\subset p(M/N)^{-(k+1)}N$, provided $p(M/N)^{-k}N\neq M$. But, $p(M/N)^{-\infty}N$ is accessible by (A). Hence, there exists a positive integer h such that $p(M/N)^{-h}N=M$, which means that $p(M/N)$ is nilpotent modulo $l(M/N)$.

(C): We have seen just above that M is a left s-module. Again, taking the validity of Lemma 3 into mind, it is enough to prove that $P(M)$ is non-empty and finite.

First, we shall show that $P(M)$ is non-empty. Suppose, on the contrary, that $P(M)$ is empty. Then, we can find a descending chain of non-zero submodules of M: $M_t \supset M_{t-1} \supset \cdots \supset M_1$ ($t > s(0)$) such that $\mathfrak{r}_t \subset \mathfrak{r}_{t-1} \subset \cdots \subset \mathfrak{r}_1$ where $\mathfrak{r}_i = p(M_i)$. We set $M'_0 = 0$, $M'_i = \mathfrak{r}_i^{-\infty} M'_{i-1}$ ($i = 1, 2, \cdots, t$), and choose a positive integer f such that $\mathfrak{r}_i^f M_i = 0$ and $M'_i = \mathfrak{r}_i^{-f} M'_{i-1}$ for all i. Evidenty, $M_i \subseteq (\mathfrak{r}_1^f \cdots \mathfrak{r}_i^f)^{-1} M'_0 = M'_i$. On the other hand, $M_{i+1} \not\subseteq M'_i$. In fact, if not, $\mathfrak{r}_1^f \cdots \mathfrak{r}_i^f M_{i+1} = 0$ implies $\mathfrak{r}_i^{fi} \subseteq \mathfrak{r}_1^f \cdots \mathfrak{r}_i^f \subseteq l(M_{i+1})$, which forces a contradiction $\mathfrak{r}_i \subseteq \mathfrak{r}_{i+1}$. We obtain therefore $M'_0 \subset M'_1 \subset \cdots \subset M'_t$. But, this is impossible.

Next, we shall prove the finiteness of $P(M)$. Let $P = \{\mathfrak{p}_\lambda \mid \lambda \in \Lambda\}$ be an arbitrary non-empty subset of $P(M)$: $\mathfrak{p}_\lambda = p(N_\lambda)$ with a left p-stable submodule N_λ of M. We consider a finite subset $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$ of P such that $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for every $i < j$. To be easily seen, we have then $N_i \not\supseteq N_j$ for every $i < j$. We set here $N'_0 = 0$, $N'_i = \mathfrak{p}_i^{-\infty} N'_{i-1}$ ($i = 1, 2, \cdots, k$), and choose a positive integer h such that $\mathfrak{p}_i^h N_i = 0$ and $N'_i = \mathfrak{p}_i^{-h} N'_{i-1}$ for all i. Then, $N_i \subseteq (\mathfrak{p}_1^h \cdots \mathfrak{p}_i^h)^{-1} N'_0 = N'_i$. On the other hand, we have $N_{i+1} \not\subseteq N'_i$. In fact, if not, $\mathfrak{p}_1^h \cdots \mathfrak{p}_i^h N_{i+1} = 0$ implies $\mathfrak{p}_1^h \cdots \mathfrak{p}_i^h \subseteq \mathfrak{p}_{i+1}$. Recalling that $\mathfrak{p}_{i+1}$ is a prime ideal (Remark 1), we obtain $\mathfrak{p}_j \subseteq \mathfrak{p}_{i+1}$ for some $j < i+1$, which is a contradiction. It follows therefore $N'_0 \subset N'_1 \subset \cdots \subset N'_k$, and so $k \leqslant s(0)$ by (A). From what we have proved just now, we see that the set of all maximal members of P is non-empty and finite. Now, let P'_1 be the set of all maximal members of $P_1 = P(M)$, and $P_2 = P_1 \backslash P'_1$. If P_2 is non-empty, we consider P'_2 the set of all maximal members of P_2 and set $P_3 = P_2 \backslash P'_2$. Repeating this procedure, we obtain the descending chain $P_1 \supset P_2 \supset P_3 \supset \cdots$. Suppose $P_{s(0)+1}$ is non-empty. Then, we can choose $\mathfrak{p}_i \in P_i$ such that $\mathfrak{p}_1 \supset \mathfrak{p}_2 \supset \cdots \supset \mathfrak{p}_{s(0)+1}$, which contradicts the remark stated above. We have proved thus $P(M)$ is finite.

Now, we can state the following theorem (cf. [3; Ths. 1.7 and 2.7], [5; Th. 11] and [6; Th. 8]):

Theorem 5. *The following conditions are equivalent*: (i) (A)+(B)+(D), (ii) (A)+(B)+(E), (iii) (C)+(D)+(F), (iv) (C)+(D′), (v) (C)+(E)+(F), *and* (vi) (G).

Proof. (vi) implies (i) – (v): (A), (B), (C), (E) and (F) are evident by Lemma 2, Th. 1 and Props. 4 and 5. Especially, if $\{N_i \mid i=1, \cdots, s\}$ is a left s-primary decomposition of a left P-submodule N then s must equal 1 (Th. 1), and so $N=N_1$ is left primary, proving (D).

(iii) implies (iv): This is trivial.

(iv) implies (vi): This is a direct consequence of [2; Th. 4. 10].

(v) implies (iii): This is contained in Lemma 1.

(i) Implies (iii) and (ii) implies (v): These are obvious by Prop. 5.

As is easily seen, [3; Lemma 2. 4 and Prop. 2. 5] are still valid for a left s-module. Combining this remark with Th. 5, we readily obtain the following, which contains [3; Th. 2. 6]:

Theorem 6. *If a left Artin-Rees module M is a left s-module whose each factor module is finite-dimensional (in the sense of Goldie [4]), then M has the left s-primary decomposition theory.*

Finally, the proof of the next proceeds in the same way as that of [3; Th. 2. 9] did.

Theorem 7. *Suppose M has the left s-primary decomposition theory. If $\mathfrak{a}$ is an ideal of R and $N=\bigcap_{n=1}^{\infty}\mathfrak{a}^n M$, then $\mathfrak{a}N=N$.*

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