ON THE SOLVABILITY OF UNIT GROUPS OF GROUP RINGS

Dedicated to Professor MASARU OSIMA on the occasion of his 60th birthday

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Let P be a 2-Sylow subgroup of GL(2, 3). Then, it is clear that $P=\langle a, b : a^8=1, b^2=1, bab^{-1}=a^3\rangle$. Since the unit group of the group ring of P over GF(3) is solvable by Lemma 1 (2), we see that Lemma 2, Theorem 11, Theorem 12, Corollary 1 and Corollary 2 of J. M. Bateman's paper [2] are all incorrect. They should be corrected respectively as in Lemma 1 (3), Theorem, Corollary, Proposition 2 and Proposition 3 of the present paper.

In what follows, G will represent always a finite group, G' the commutator subgroup of G, G_p a p-Sylow subgroup of G, and $O_p(G)$ the maximal normal p-subgroup of G. Moreover, R will denote an Artinian simple ring with center C, RG the group ring of G over R, $U(RG)$ the unit group of RG, and $J(RG)$ the radical of RG. We set $D=\langle a, b : a^4=1, b^2=1, bab^{-1}=a^{-1}\rangle$, $Q=\langle a, b : a^4=1, a^2=b^2, bab^{-1}=a^{-1}\rangle$ and $P=\langle a, b : a^8=1, b^2=1, bab^{-1}=a^3\rangle$. We notice that a result of P. B. Bhattacharya and S. K. Jain [3 ; Theorem 1] will be used freely.

Lemma 1. *Let R be* GF(3). *Then there holds the following*:

(1) *$U(RA_4)$ is not solvable, where A_4 is the alternating group of degree* 4.

(2) *If $G=D, Q$ or P, then $U(RG)$ is solvable.*

(3) *If H is a non-abelian subgroup of* GL(2, 3) *such that $U(RH)$ is solvable and H' is not a* 3*-group, then H is isomorphic to D, Q or P.*

Proof. (1) A_4 has an absolutely irreducible representation of degree 3 over the rational number field and hence, by [4 ; Theorem 1], A_4 has also an absolutely irreducible R-representation of degree 3. Hence, $U(RA_4)$ is not solvable.

(2) Since the factor group G/G' is a 2-elementary abelian group of order 4, G has four linear R-representations. Concerning non-linear irreducible representations, it is known that D has one absolutely irreducible (faithful) R-representation U:

$$U(a)=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad U(b)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Q has one absolutely irreducible (faithful) R-representation V:

$$V(a)=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad V(b)=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and that P has two absolutely irreducible (faithful) R-representations T_1, T_2 and one absoultely irreducible (non-faithful) R-representation T_3:

$$T_1(a)=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad T_1(b)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$T_2(a)=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad T_2(b)=\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$T_3(a)=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T_3(b)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence, $U(RG)$ is solvable in either case.

(3) Since $|\mathrm{GL}(2, 3)|=48$, $|H|=6, 8, 12, 16, 24$ or 48. If $|H|=6$ then H' is a 3-group, because H is isomorphic to the symmetric group of degree 3. If $|H|=8$, then

$$H\cong\left\langle\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle\cong D$$

or

$$H\cong\left\langle\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\rangle\cong Q.$$

If $|H|=12$ then H' is a 3-group. In fact, every subgroup of order 12 of $\mathrm{GL}(2, 3)$ is dihedral. If $|H|=16$ then H is a 2-Sylow subgroup of $\mathrm{GL}(2, 3)$, and hence $H\cong P$. If $|H|=24$ then $H=\mathrm{SL}(2, 3)$. Since $\mathrm{PSL}(2, 3)\cong A_4$ ([1; p. 170]), $U(RH)$ is not solvable by (1). Finally, if $|H|=48$ then $U(RH)$ is not solvable by $H=\mathrm{GL}(2, 3)\supset\mathrm{SL}(2, 3)$.

Lemma 2. *If $R=\mathrm{GF}(3)$ and $U(RG)$ is solvable, then G_3 is normal and G/G_3 is either an abelian group or a non-abelian 2-group.*

Proof. Suppose that $G^*=G/O_3(G)$ is non-abelian. Then, by [2; Theorems 7 (1)] and the solvability of $U(RG)$, $G^*\subseteq U(RG)/1+J(RG)\cong U(RG/J(RG))\cong U(R(G/G'G_3))\times\mathrm{GL}(2, 3)^{(s)}$ for some positive integer s. Hence, G^* is isomorphic to a subgroup of $\Pi_{i\in I_1\cup I_2} T_i(G)$, where $\{T_i : i\in I_1\}$ is the complete set of inequivalent irreducible R-representations of G such that $T_i(G)$ is commutative and $\{T_i : i\in I_2\}$ is the complete set of inequivalent irreducible R-representations of G such that $T_i(G)$ is non-commutative. Since every $T_i(G)$ $(i\in I_2)$ is a subgroup of GL(2, 3) such that $U(RT_i(G))$ is solvable and $T_i(G)'$ is not a 3-group, we see that G^* is a subgroup of a direct product of cyclic groups of order relatively prime to 3 and copies of D, Q and P (Lemma 1 (3)). Therefore, G_3 is a normal subgroup of G. Further, $U(R(G/G_3))$ being solvable, G/G_3 is a non-abelian 2-group by [5 ; Theorem 1].

In case G is a 2-group, we set $e(G)=(|G|-(G: G'))/4$, $r(G)=|\{N\lhd G: G/N\cong P\}|$ and $s(G)=|\{N\lhd G: G/N\cong D$ or $Q\}|$. Then, recalling that $\{T_1, T_2\}$, U and V are absolutely irredubible faithful GF(3)-representations of P, D and Q, respectively (cf. the proof of Lemma 1 (2)), we can see that G has at least $2r(G)+s(G)$ absolutely irreducible GF(3)-representations of degree 2.

Proposition 1. *Let R be* GF(3), *and G a 2-group. Then, $U(RG)$ is solvable if and only if $e(G)=2r(G)+s(G)$.*

Proof. If $U(RG)$ is solvable, then a brief computation shows that $RG\cong R(G/G')\oplus(R)_2^{(e(G))}$. Accordingly, by the above remark, it follows $e(G)\geqq 2r(G)+s(G)$. Further, if T is an arbitrary irreducible representation of degree 2 induced by a simple component of capacity 2 then $G/\mathrm{Ker}\, T$ is isomorphic to one of D, Q and P (Lemma 1 (3)), which means $e(G)\leqq 2r(G)+s(G)$. Conversely, assume that $e(G)=2r(G)+s(G)$. Then, we can easily see that $RG=R(G/G')\oplus(R)_2^{(e(G))}$, namely, $U(RG)$ is solvable.

Now, we can prove the following theorem, which contains [5 ; Theorem 1].

Theorem. *$U(RG)$ is solvable if and only if there holds one of the following*:

(1) *$R=C$ (of characteristic p), G_p is normal, and G is a semi-direct product of G_p and an abelian group.* ($G_0=1$ *by definition.*)

(2) $R=$GF(2), *G_3^* is normal and elementary abelian, and G^* is a semi-direct product of G_3^* and a group $\langle a\rangle$ of order 2 such that $axa^{-1}=x^{-1}$ for every $x\in G_3^*$, where $G^*=G/O_2(G)$.*

(3) $R=$GF(3), *G_3 is normal, $e(G_2)=2r(G_2)+s(G_2)$, and G is a*

semi-direct product of G_3 *and* G_2.

(4) $R=(\mathrm{GF}(2))_2$ *and* $G=G_2$.

(5) $R=(\mathrm{GF}(3))_2$, G_3 *is normal,* G_2 *is elementary abelian, and* G *is a semi-direct product of* G_3 *and* G_2.

Proof. If $U(RG)$ is solvable then $U(R)$ is solvable, and hence $R=C$, $(\mathrm{GF}(2))_2$ or $(\mathrm{GF}(3))_2$. We shall distinguish therefore between these three cases.

Case 1. $R=C\neq \mathrm{GF}(2)$ or $\mathrm{GF}(3)$: Let R be of characteristic p. Then, $G/O_p(G)$ is abelian and hence G_p is normal and G is a semi-direct product of G_p and an abelian group. ($O_0(G)=1$ *by difinition.*)

Case 2. $R=\mathrm{GF}(2)$ or $\mathrm{GF}(3)$: If $R=\mathrm{GF}(2)$ (resp. $\mathrm{GF}(3)$), then we may restrict our attention to the case where $G/O_2(G)$ (resp. $G/O_3(G)$) is non-abelian. Now, (2) (resp. (3)) is clear by [2; Theorems 9 and 10][1] (resp. Lemma 2 and Proposition 1).

Case 3. $R=(\mathrm{GF}(2))_2$ or $(\mathrm{GF}(3))_2$: Since $RG/J(RG)$ is isomorphic to $(CG/J(CG))_2$ and $U(RG/J(RG))$ is solvable, we obtain $CG/J(CG)=C^{(t)}$ for some positive integer t. Further, we claim that $G/O_p(G)$ is ismorphic to a subgroup of $U(GF(p)G/J(GF(p)G))$ ([2; Theorem 7 (i)]). If $C=\mathrm{GF}(2)$, then $G/O_2(G)=1$, namely, $G=G_2$. While, if $C=\mathrm{GF}(3)$ then $G/O_3(G)$ is 2-elementary abelian, so that G_3 is normal and G is a semi-direct product of G_3 and an elementary abelian 2-Sylow subgroup.

The proof of the converse is obvious by [5; Theorems 1 and 2], [2; Theorem 9] and Proposition 1.

Corollary. *Let* G *be a non-abelian 2-group, and* $R=\mathrm{GF}(3)$. *If* $U(RG)$ *is solvable then* G *is a subdirect product of copies of* Z_2, Z_4, D, Q *and* P, *where* Z_i *is a cyclic group of order* i.

Proof. As is easily seen from the proof of Lemma 2, G is a subdirect product of cyclic 2-groups and copies of D, Q and P. If G has an absolutely irreducible R-representalion T such that $T(G)\cong P$, then G has also an absolutely irreducible R-representation T' such that $T'(G)\cong D$ (cf. the proof of Lemma 1 (2)). Accordingly, it suffices to prove that if S is a subdirect product of Z_8 and one of D and Q then $U(RS)$ is not solvable.

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1) In the proof of [2; Theorem 10], $\rho_1,\cdots,\rho_s$ and $\rho_1,\cdots,\rho_l$ should be understood respectively as the irreducible representatins of G over F and the commutative irreducible representations.

First, assume that S is a subdirect product of $Z_8=\langle c\rangle$ and D. Then, up to isomorphism, $S=\langle c\rangle\times D$, $S=\langle c\rangle*D=\langle c^2\rangle\times\langle a\rangle\cup c\langle c^2\rangle\times b\langle a\rangle$ or $S=\langle c\rangle**D=\langle c^2\rangle\times\langle a^2, b\rangle\cup c\langle c^2\rangle\times a\langle a^2, b\rangle$. If $S=\langle c\rangle\times D$, then $U(RS)$ is not solvable by [2; Lemm 3]. If $S=\langle c\rangle*D$, then $|S|=32$, $|S'|=2$ and $e(S)=4$. S contains only two normal subgroups $N_1=\langle(c^2, 1)\rangle$ and $N_2=\langle(c^2, a^2)\rangle$ of index 8 not containing S', and does not contain a normal subgroup N such that $S/N\cong P$. Hence, $U(RS)$ is not solvable by Theorem (or Proposition 1). If $S=\langle c\rangle**D$, then $|S|=32$, $|S'|=2$ and $e(S)=4$. Again S contains only two normal subgroups $N_1=\langle(c^2, 1)\rangle$ and $N_2=\langle(c^2, a^2)\rangle$ of index 8 not containing S', and does not contain a normal subgroup N such that $S/N\cong P$. Hence, $U(RS)$ is not solvable by Theorem.

Next, assume that S is a subdirect product of $Z_8=\langle c\rangle$ and Q. Then, up to isomorphism, $S=\langle c\rangle\times Q$ or $S=\langle c\rangle*Q=\langle c^2\rangle\times\langle a\rangle\cup c\langle c^2\rangle\times b\langle a\rangle$. If $S=\langle c\rangle\times Q$, then $U(RS)$ is not solvable by [2; Lemma 3]. If $S=\langle c\rangle*Q$, then $|S|=32$, $|S'|=2$ and $e(S)=4$. S contains only two normal subgroups $N_1=\langle(c^2, 1)\rangle$ and $N_2=\langle(c^2, a^2)\rangle$ of index 8 not containing S', and does not contain a normal subgroup N such that $S/N\cong P$. Hence, $U(RS)$ is not solvable again by Theorem.

The next is clear by the preceding corollary.

Proposition 2. *Let G be a non-abelian 2-group, and $R=$*GF(3). *If $U(RG)$ is solvable then G is of exponent 8, the center $Z(G)$ of G is of exponent 2, and $G/Z(G)$ is of exponent 4.*

Finally, Proposition 2 enables us to apply the same argument as in the proof of [2; Lemma 3 and Corollary 2] to see the following:

Proposition 3. *Let G be a non-abelian 2-group, and $R=$*GF(3). *If $U(RG)$ is solvable, then $G=E\times I$, where E is 2-elementary abelian and I is an indecomposable non-abelian 2-group which is a subdirect product of copies of Z_4, Q, D and P.*

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(Received April 28, 1972)