MONOTONE COMPLETENESS AND CONVERGENCE THEOREM

Dedicated to Professor TAKESHI INAGAKI on his 60th birthday

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1. Introduction

In this note, we shall show that L. Schwartz's convergence theorem [5] is still valid, when the linear topology of a sequential linear topological space of measurable functions is monotone complete and continuous. For this purpose, we shall consider "monotone completeness" and discuss some related topics, though substantial investigations on the theme have been done in [2], [3] and [4].

2. Sequential topology

Let Ω be a measure space with measure μ. We shall assume that for every non-null measurable set A of Ω, $\mu(A)=\sup_{F\subset A}\mu(F)$ where F is a measurable set of non-null finite measure. Let L be a linear space of μ-measurable functions which take finite values for almost all Ω with respect to μ. We shall suppose that L is *normal*, i. e. if $x(t)\in L$ and $y(t)$ is measurable with $|y(t)|\leqq|x(t)|$ a. e. for $t\in\Omega$, then $y(t)\in L$. In the sequel, we shall identify two elements of L if they coincide except on a measure zero set. A linear topology τ on L is called an *ordered topology* (*order-preserving topology*) if there exists a base of neighborhood system of 0 which is normal, i. e. there exists a neighborhood system of 0 : $\{V_\lambda\}_{\lambda\in\Lambda}$ such that $x(t)\in V_\lambda$, $y(t)\in L$ with $|y(t)|\leqq|x(t)|$ imply $y(t)\in V_\lambda$. A functional $\|\cdot\|$ defined on L is called a *quasi-norm* if the following conditions are satisfied :

(1) $\|x\|=0\Leftrightarrow x=0,$

(2) $|x|\leqq|y|\Rightarrow\|x\|\leqq\|y\|,$

(3) $\|\xi x\|\to 0$ if $\xi\to 0$ for a fixed $x\in L,$

(4) $\|\xi x\|\to 0$ if $\|x\|\to 0$ for a fixed $\xi,$

(5) $\|x+y\|\leqq\|x\|+\|y\|$ for $x,\ y\in L.$

If $V_n=\{x;\ \|x\|\leqq 1/2^n\}$, then $\{V_n\}$ $(n=1, 2, \cdots)$ defines an ordered sequential topology on L.

Theorem 1. *An ordered linear topology τ on L is sequential iff there exists a quasi-norm $\|\cdot\|$ and τ is equivalent to the topology defined by $\|\cdot\|$.*

Proof. Let $\{V_n\}$ $(n=1, 2, \cdots)$ be a base of neighborhood system of 0, where V_n is normal, symmetric and $V_{n+1}+V_{n+1}\subset V_n$ for all $n\geqq 1$.

$$U_q=V_1^{p_1}+V_2^{p_2}+\cdots+V_k^{p_k}$$

where $q=\sum_{n=1}^{k}(p_n/2^n)$ and $p_n=0$ or 1 and $V_n^1=V_n$, $V_n^0=\{0\}$.

Putting

$$\|x\|=\inf_{x\in U_q} q,$$

then we have

$$\{x:\|x\|<1/2^n\}\subset V_n\subset\{x:\|x\|\leqq 1/2^n\}$$

and $\|\cdot\|$ satisfies the conditions of quasi-norm.

We shall consider some properties concerning the topology and order in L.

A normal subset $A\subset L$ is called semi-continuous if $0\leqq x_i\uparrow_i x$, $x_i\in A$ $(i=1, 2, \cdots)$ imply $x\in A$.

An ordered topology τ in L is called *semi-continuous* if there exists a base of neighborhood system $\{V_\lambda\}_{\lambda\in\Lambda}$ which are semi-continuous. We have the following theorem by definition and by Theorem 1.

Theorem 2. *A sequential order-topology τ on L is semi-continuous iff there exists a quasi-norm which is equivalent to τ such that*

$$0\leqq x_i\uparrow_i x \textit{ implies } \sup_i \|x_i\|=\|x\|.$$

An ordered topology τ on L is called *quasi-continuous* if there exists a base of normal neighborhood system $\{V_\lambda\}_{\lambda\in\Lambda}$ of 0 such that for every V_λ, there exist $\alpha>0$ and $\lambda'\in\Lambda$ such that $0\leqq x_i\uparrow_i x$, $x_i\in V_{\lambda'}$ imply $x\in\alpha V_\lambda$. A topology τ on L is *continuous* if $x_i\downarrow_i 0$ implies $x_i\to 0$ by τ. If τ is continuous, $0\leqq x_i\uparrow_i x$ implies $x_i\to x$ by τ. If τ is continuous, then τ is semi-continuous and if τ is semi-continuous, then τ is quasi-continuous. By definition, τ is *monotone-complete* if every topologically bounded monotone countable set A is order-bounded, i. e. if $0\leqq x_i\uparrow_i$ and $A=\{x_i\}$

is topologically bounded, then there exists x such that $0 \leqq x_i \uparrow_i x$ (cf. [3]).

Theorem 3. *If a sequential ordered topology τ on L is monotone complete, then τ is complete.*

Proof. We shall show that τ is quasi-continuous if τ is monotone complete. There exists a normal neighborhood system of zero V_n such that $V_{n+1}+V_{n+1} \subset V_n$ and $\bigcap_n V_n = \{0\}$. Let

$$\Gamma(V_n) = \{x;\ 0 \leqq x_i \uparrow_i x \text{ for some } x_i \in V_n\}.$$

We shall show that for every V_n, there exists $\alpha > 0$ with $\alpha V_n \supset \Gamma(V_m)$ for some m. If not, for every integer k there exist n_k and x_k with

$$x_k \notin kV_n,\ \ x_k \in \Gamma(V_{n_k})$$

and there exist $0 \leqq x_{k,i} \uparrow_i x_k$ with $x_{k,i} \in V_{n_k}$ and $n < n_1 < n_2 < \cdots$.

Put

$$b_k = x_{1,k} + x_{2,k} + \cdots + x_{k,k}\ (k \geqq 1).$$

Then $b_k \in V_{n_1} + \cdots + V_{n_k}$ and $\{b_k\}$ is topologically bounded. Hence, there exists $b \in L$ such that $0 \leqq b_k \uparrow_k b$ and $b \notin kV_n$ $(k=1, 2, \cdots)$. Since a set consisting of a single element b is automatically topologically bounded, this is a contradiction. Hence, τ is quasi-continuous.

Let now $\{x_i\}$ be a Cauchy sequence by τ. By taking partial sequence, we find $n_1 < n_2 < \cdots$ with $\|x_{n_i} - x_{n_{i+1}}\| < 1/2^i$. Putting $b_1 = x_{n_1}$ and $b_i = x_{n_i} - x_{n_{i-1}}$ for $i \geqq 2$, we have

$|b_i| + |b_{i+1}| + \cdots + |b_{i+n}| \in V_{i-1} = \{x;\ \|x\| \leqq 1/2^{i-1}\}$. Hence, $\{|b_1| + |b_2| + \cdots + |b_n|\}$ $(n=1, 2, \cdots)$ is topologically bounded, so is order bounded by the monotone completeness of τ. Hence, $\sum_{i=1}^{k} |b_i|$ is order convergent to $\sum_{i=1}^{\infty} |b_i|$, and so $x_{n_k} = \sum_{i=1}^{k} b_i$ $(k=1, 2, \cdots)$ is order convergent to $b = \sum_{i=1}^{\infty} b_i$ (cf. Theorem 3.4 in [3]). By quasi-continuity, for every m, there exists N such that $|x_{n_k} - b| \leqq \sum_{i=k+1}^{\infty} |b_i| \in V_m$ for $k \geqq N$, since $\sum_{i=k+1}^{m} |b_i| \in V_k$ for $m \geqq k+1$. It means that $x_{n_k} = \sum_{i=1}^{k} b_i$ converges to b by the topology τ. Since apartial sequence x_{n_k} converges, x_n converges to b, therefore τ is complete.

3. Completion of topological vector lattices

In this section, we shall consider the completion of a topological vector lattice. Let L be a σ-complete vector lattice with linear topology τ in which there exists a countable normal neighborhood system of 0. Let $\tilde{L}$ be the completion of L with respect to τ. We shall define an order relation in $\tilde{L}$. $\tilde{L}$ is considered as a set of equivalent classes of Cauchy sequences of L, since τ is sequential.

We define $\tilde{x} \geqq \tilde{y}$ for $\tilde{x}, \tilde{y} \in \tilde{L}$ if there exist Cauchy sequences $\{x_i\} \in \tilde{x}$, $\{y_i\} \in \tilde{y}$ with $x_i \geqq y_i (i=1, 2, \cdots)$ such that $x_i \in L$, $y_i \in L$.

Lemma 1. *$\{x_i\} \in \tilde{x}$, $\{x'_i\} \in \tilde{x}$ imply $\{x_i \cup x'_i\}$, $\{x_i \cap x'_i\} \in \tilde{x}$.*

Proof. $|x_i \cup x'_i - x_j \cup x'_j| \leqq |x_i \cup x'_i - x_j \cup x'_i| + |x_j \cup x'_i - x_j \cup x'_j| \leqq |x_i - x_j| + |x'_i - x'_j|$. Hence, $\{x_i \cup x'_i\}$ is a Cauchy sequence. Similarly we see that $\{x_i \cap x'_i\}$ is a Cauchy Sequence. Since $|x_i \cup x'_i - x_i| = |0 \cup (x'_i - x_i)| \leqq |x'_i - x_i|$ and $|x_i \cap x'_i - x_i| = |0 \cap (x'_i - x_i)| \leqq |x'_i - x_i|$, the Cauchy sequences $\{x_i \cup x'_i\}$ and $\{x_i \cap x'_i\}$ are equivalent to the Cauchy sequence $\{x_i\}$.

Lemma 2. *The relation $\geqq$ in $\tilde{L}$ is an order relation:*

(1) $\tilde{x} \geqq \tilde{x}$,
(2) $\tilde{x} \geqq \tilde{y}$ *and* $\tilde{y} \geqq \tilde{x}$ *imply* $\tilde{x} = \tilde{y}$,
(3) $\tilde{x} \geqq \tilde{y}$ *and* $\tilde{y} \geqq \tilde{z}$ *imply* $\tilde{x} \geqq \tilde{z}$.

Proof. (1) is clear from definition. (2) Let $\{x_i\} \in \tilde{x}$, $\{y_i\} \in \tilde{y}$ with $x_i \geqq y_i$, and $\{x'_i\} \in \tilde{x}$, $\{y'_i\} \in \tilde{y}$ with $y'_i \geqq x'_i$. Then by Lemma 1

$$x_i \geqq x_i \cap y'_i \geqq x_i \cap x'_i \sim x_i, \text{ and}$$
$$y'_i \geqq x_i \cap y'_i \geqq y_i \cap y'_i \sim y'_i$$

where $\sim$ means the equivalent relation for Cauchy sequences. Hence, $\{x_i\} \sim \{x_i \cap y'_i\} \sim \{y'_i\}$. This proves (2).

Let $\tilde{x} \geqq \tilde{y}$ and $\tilde{y} \geqq \tilde{z}$. Then we have $x_i \geqq y_i$, $y'_i \geqq z_i$ for some $\{x_i\} \in \tilde{x}$, $\{y_i\}$, $\{y'_i\} \in \tilde{y}$ and $\{z_i\} \in \tilde{z}$. Since $\{x_i\} \sim \{x_i \cup y'_i\}$ and $x_i \cup y'_i \geqq y'_i \geqq z_i$, we have $\tilde{x} \geqq \tilde{z}$.

Proposition 1. *$\tilde{L}$ is an ordered topological vector lattice with sequential linear topology $\tilde{\tau}$ induced by τ.*

Proof. We have only to prove that there exists $\tilde{x} \cup 0$ for $\tilde{x} \in \tilde{L}$. We see easily that $\{x_i \cup 0\}$ is a Cauchy sequence if $\{x_i\}$ is a Cauchy sequence. We shall denote this Cauchy sequence by $\tilde{x}_0$. Let $\tilde{y} \geqq \tilde{x}$ and $\tilde{y} \geqq 0$. Then, for some $\{x_i\} \in \tilde{x}$ and $\{y_i\}$, $\{y'_i\} \in \tilde{y}$ we have $y_i \geqq x_i$ and $y'_i \geqq 0$ and $\{y_i \cup y'_i\} \sim \{y_i\} \sim \{y'_i\}$. Since $y_i \cup y'_i \geqq x_i \cup 0$, we have $\tilde{y} \geqq \tilde{x}_0$. Hence $\tilde{L}$ is a vector lattice.

4. Topological conditions

We shall consider the following condition :

(A) If $0 \leqq x_i \uparrow$ and $\{x_i\}$ is topologically bounded, then $\{x_i\}$ is a Cauchy sequence by τ.

If τ satisfies the above condition, then τ is called *s-coninuous.*

Theorem 4. *If τ is s-contiuuons and quasi-continuous, then τ is continuous.*

Poof. If $0 \leqq x_i \uparrow_i x$, then $\{x_i\}$ is a Cauchy sequence by s-continuity. For every V_n, there exists V_m and a number $N(n)$ such that $x_j - x_i \in V_m$ for $j \geqq i \geqq N(n)$ and $0 \leqq x - x_i \in V_n$ by quasi-continuity. Hence, x_i converges to x by τ.

By definition, we have

Theorem 5. *Let τ be complete. τ is s-continuous iff τ is monotone-complete and continuous.*

Theorem 6. *τ is s-continuous iff τ satisfies the following condition :*

(*) *If $0 \leqq x_i \uparrow_i$, $x_i - x_{i-1} \perp x_{i-1}$ and $\{x_i\}$ is topologically bounded by τ, then $\{x_i\}$ is a Cauchy sequence by the topology τ.*

Proof. We have only to prove that τ is s-continuous from (*). Let $0 \leqq x_i \uparrow_i$ and $\{x_i\}$ be topologically bounded. Then, $\lim_{i\to\infty} x_i(t) = x(t)$ a. e. and $x(t)$ is finite a. e.

Since (*) is satisfied, for every $x \in L$, $\{t\,;\, x(t) \neq 0\}$ is a countable union of measurable sets of finite measure. Hence, as we easily see, for every $n \geqq 2$, there exist a number $i(n)$ and a measurable set $M_n \subset \{t\,;\, x_{i(n)}(t) \geqq (1-1/n)x(t)\}$ with $i(2) < \cdots < i(n) < \cdots$ such that $M_2 \subset M_3 \subset \cdots$ with $\cup M_n = \{t\,;\, x(t) \neq 0\}$. Putting $b_n = \chi_{M_n} \cdot x$ where χ_{M_n} is the characteristic function of M_n, we have $0 \leqq b_n \leqq \{1/(1-1/n)\} x_{i(n)}$ and $b_n \in L$.

Since $\{x_i\}$ is topologically bounded, $\{b_n\}$ is topologically bounded, therefore $\{b_n\}$ is a Cauchy sequence by (*). Now, we shall consider the completion $\tilde{L}$ of L. $\tilde{L}$ is a vector lattice, as we have shown already. For the limiting element $b \in \tilde{L}$ of $\{b_n\}$, we have $0 \leqq b_n \leqq b$ (the order being considered in $\tilde{L}$) and $\|b_n - b\| \leqq \varepsilon$ for a prescribed $\varepsilon > 0$ and for $n \geqq n(\varepsilon)$ where $n(\varepsilon)$ is a number determined by ε. Hence for $m \geqq N$,

$0 \leqq x_{i(m)} - x_{i(n)} \leqq (1/(1-1/m)b_m - (1-1/n)b_n + 2(b-b_n)$, $\| x_{i(m)} - x_{i(n)} \|$
$\leqq \| (1/(1-1/m))\, b_m - (1-1/n)b_n \| + \| 2(b-b_n) \| \to 0$ as $n, m \to \infty$.
Hence, $\{x_n\}$ is a Cauchy sequence.

5. Convergence of C-sequencc

In this section, we shall consider a generalization of L. Schwartz's theorem.

Theorem 7. *Let τ be a sequential order topology on L, complete and continuous, and let $x_n \to 0\ (\tau)$. Then, on every finite measurable set $x_n \to 0$ (in measure).*

Proof. Suppose that $\mu(t\,;\ |x_n(t)| \geqq \varepsilon,\ t \in M) \geqq \delta > 0$ and $\| x_n \| \leqq 1/2^n$, where M is a finite measurable set. By the completeness of τ, there exists $x = \sum_{n=1}^{\infty} x_n \in L$. Similarly, there exists $y_n = \sum_{k=n}^{\infty} |x_k| \in L$ for all $n = 1, 2, \cdots$ and it is easily shown that $\| y_n \| \to 0$. Let $E_n = \{t\,;\ y_n(t) \geqq \varepsilon\} \cap M$. Then $\mu(E_n) \geqq \delta$. On the other hand, we find y_0 such that $y_n \downarrow y_0 \in L$. For a measurable set $E = \bigcap_{n=1}^{\infty} E_n \subset \{t\,;\ y_0(t) \geqq \varepsilon\}$, we have $\mu(E) \geqq \delta$, i. e. $y_0 \neq 0$. But, this is a contradiction by the continuity of τ.

If for every $\{c_n\} \in c_0$ (i. e. sequence of real numbers with limit zero), $\sum_n c_n x_n$ is convergent by τ, then $\{x_n\}$ is called a *C-sequence* by τ.

Theorem 8. *Let τ be monotone-complete and continuous. Then, for every C-sequence $\{x_n\}$, x_n converges to 0 by the topology τ.*

Proof. By Kolmogorov-Khintchine's theorem (i. e. for every C-sequence $\{x_n\}$ by the topology of convergence in measure almost everywhere on every finite mesurable set, $x_n \to 0$ (a. e.) on every finite measurable set, cf. [1]), $x_n \to 0$ (a. e.) on every finite measurable set, since $\{x_n\}$ is a C-sequence by the topology of convergence in measure by Theorem 7.

Suppose that the sequence x_n does not converge to 0 by the topology τ. We shall show that there exist pairwise disjoint finite measurable sets $M_k (k=1, 2, \cdots)$ and $n_1 < n_2 < \cdots < n_k < \cdots$ such that

(1) $\| \chi_{M_n} x_{n_k} \| > \delta$ for some $\delta > 0$,

(2) $\sum_{k=1}^{\infty} \sum_{j \neq k} \| \chi_{M_k} x_{n_j} \| < \infty.$

We can find a positive number δ such that

(3) $\qquad \|x_n\| \geqq 3\delta \qquad$ for infinitely many n.

By induction, let $n_1 < n_2 < \cdots < n_k$, and finite measurable sets $K_1, K_2, \cdots, K_k$ be chosen so that

(4) $$\|\chi_{K_j} x_{n_j}\| \geqq 2\delta \qquad (j=1, \cdots, k)$$

and

(5) $$\|\chi_{K_k} x_{n_j}\| \leqq \frac{1}{2^{k+j}}\delta \qquad (j=1, \cdots, k-1).$$

Let $x \in L$ and $x_n \to 0$ (a. e.) on a finite measurable set M. Then, by Egoroff's theorem and the continuity of τ, we see that for every $\varepsilon > 0$ there exists a measurable set $S \subset M$ with

$$\|\chi_S x\| \leqq \varepsilon$$

and

$$\|\chi_{M-S} x_n\| \to 0 \text{ an } n \to \infty.$$

Hence, applying this to a finite measurable set $M = T_k \supset K_1 \cup K_2 \cup \cdots \cup K_k$ and $x = |x_{n_1}| + \cdots + |x_{n_k}|$, we can find a finite measurable set S_k such that

(6) $$\|\chi_{T_k - S_k} x_n\| \leqq \frac{1}{2^{2k+1}}\delta \qquad \text{for sufficiently large } n$$

and

(7) $$\|\chi_{S_k} x_{n_j}\| \leqq \frac{1}{2^{2k+2}}\delta \qquad (j=1, 2, \cdots, k).$$

We take $T_k \supset K_1 \cup K_2 \cup \cdots \cup K_k$ with the following property:

(8) $$\|\chi_{\Omega - T_k} x_{n_j}\| \leqq \frac{1}{2^{2k+2}}\delta \qquad (j=1, \cdots, k).$$

We choose $x_{n_{k+1}}$ satisfying (3) and (6) with $n_k < n_{k+1}$, i. e.

(9) $$\|\chi_{T_k - S_k} x_{n_{k+1}}\| \leqq \frac{1}{2^{2k+1}}\delta \quad \text{and} \quad \|x_{n_{k+1}}\| \geqq 3\delta.$$

Since $\|x_{n_{k+1}} - \chi_{T_k - S_k} x_{n_{k+1}}\| \geqq \|x_{n_{k+1}}\| - \|\chi_{T_k - S_k} x_{n_{k+1}}\| \geqq 3\delta - \frac{1}{2^{2k+1}}\delta > 2\delta$, by the continuity of τ we find a finite mesurable set K_{k+1} with

(10) $$\|\chi_{K_{k+1}} x_{n_{k+1}}\| \geqq 2\delta$$

and

(11) $$K_{k+1} \cap (T_k - S_k) = \emptyset.$$

Since $\|\chi_{K_{k+1}}x_{n_j}\| \leqq \|\chi_{\Omega-T_k}x_{n_j}\| + \|\chi_{S_k}x_{n_j}\| \leqq \frac{1}{2^{2k+1}}\delta$ $(j=1, \cdots, k)$ by (7) and (8), we have thus a sequence $\{K_n\}$ satisfying (4) and (5).

Let $M_k = K_k - \bigcup_{j=k}^{\infty} S_j$ $(k=1, 2, \cdots)$. $\{M_k\}$ is a sequence of pairwise disjoint finite mesurable sets with

$$\|\chi_{M_k}x_{n_k}\| \geqq \|\chi_{K_k}x_{n_k}\| - \sum_{j=k}^{\infty} \|\chi_{S_j}x_{n_k}\| \geqq 2\delta - \sum_{j=k}^{\infty} \frac{1}{2^{2j+2}}\delta > \delta \ (k=1, 2, \cdots).$$

Thus (1) is satisfied.

We shall now show that (2) is satisfied.

If $j < k$, by (5)

$$\|\chi_{M_k}x_{n_j}\| \leqq \|\chi_{K_k}x_{n_j}\| \leqq \frac{1}{2^{k+j}}\delta.$$

If $j > k$, by (9)

$$\|\chi_{M_k}x_{n_j}\| \leqq \|\chi_{T_{j-1}-S_{j-1}}x_{n_j}\| \leqq \frac{1}{2^{2j-1}}\delta \leqq \frac{1}{2^{k+j}}\delta.$$

Thus, we have (2), namely,

$$\sum_{k=1}^{\infty} \sum_{j \neq k} \|\chi_{M_k}x_{n_j}\| \leqq 2\delta.$$

Since $\{x_{n_k}\}$ is a C-sequence, $\{\chi_M x_{n_k}\}$ is a C-sequence for $M = \bigcup_{k=1}^{\infty} M_k$, and so $\{\chi_{M_k}x_{n_k}\}$ is a C-sequence by (2). We shall show however that there exists a real number sequence $c_k \to 0$ with the property: $\sum c_k y_k$ (with $y_k = \chi_{M_k}x_{n_k}$) does not converge by the topology τ.

Let us define $h(k)$ $(k=1, 2, \cdots)$ by

(12) $$h(k) = \inf_n \sup_{j \geqq 1} \|(1/k)(y_{n+1} + \cdots + y_{n+j})\|.$$

Then, $h(k)$ is a decreasing sequence of numbers. We shall show that $\lim_{k\to\infty} h(k) = \delta_1 > 0$. If $\lim_{k\to\infty} h(k) = 0$, the sequence $\sum_{k=1}^{n} y_k$ is topologically bounded, since $|y_m + \cdots + y_{m+j}| = |y_m| + \cdots + |y_{m+j}|$ and $\|y_m + \cdots + y_{m+j}\| \leqq \|y_m + \cdots + y_{m+j'}\|$ if $j < j'$ by the mutual orthogonality of $\{y_n\}$. By the s-continuity of τ, we must have $\|y_k\| \to 0$ as $k \to \infty$; this contradicts (1).

Hence, there exists $m_1 < m_2 < \cdots < m_k < \cdots$ such that

$$\|(1/k)(y_{m_k+1} + \cdots + y_{m_{k+1}})\| \geqq \delta_1/2 > 0.$$

Thus, we see that $\sum c_n y_n$ is not convergent by the topology τ for

$$c_n = \begin{cases} 1/k & \text{for } n = m_k + 1, \cdots, m_{k+1} \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof.

In his paper [5], L. Schwartz proved the following: For every C-sequence $\{x_n\}$ in L_p $(0 < p < +\infty)$, $\sum_{n=1}^{\infty} x_n$ converges by L_p-quasi-norm. The topology of L_p $(0 < p < +\infty)$ is monotone-complete and continuous. The next is a generalization of L. Schwartz's theorem.

Theorem 9. *Let τ be monotone-complete and continuous. If $\{x_n\}$ is a C-sequence by τ, then $\sum_{n=1}^{\infty} x_n$ converges by τ.*

Proof. By the above theorem, we see that $x_n \to 0$ by the topology τ. Since τ is complete, we see that $\sum_{n=1}^{\infty} x_n$ converges by the topology τ by the following lemma which is obtained by L. Schwartz:

Lemma 3. *Let L be a complete topological linear space with topology τ and $\{x_i\}$ be a C-sequence. If $x_n \to 0$ by the topology τ, then $\sum_{n=1}^{\infty} x_n$ converges by τ.*

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