

NOTE ON RESULTS OF K. MOTOSE

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Throughout the present note, R will represent a ring with 1, and \bar{R} the residue class ring of R modulo its (Jacobson) radical $J(R)$. Further, G will represent a group, and RG the group ring of G over R . Occasionally, we consider the following ring epimorphisms:

$$\begin{aligned}\phi_G: RG &\longrightarrow R \quad (\sum_{\sigma \in G} r_\sigma \sigma \longmapsto \sum_{\sigma \in G} r_\sigma), \\ \psi_G: RG &\longrightarrow \bar{R} \quad (\sum_{\sigma \in G} r_\sigma \sigma \longmapsto \sum_{\sigma \in G} \bar{r}_\sigma).\end{aligned}$$

More generally, if H is a normal subgroup and $G^* = G/H$ then we can consider the following ring epimorphisms:

$$\begin{aligned}\phi_G^H: RG &\longrightarrow RG^* \quad (\sum_{\sigma \in G} r_\sigma \sigma \longmapsto \sum_{\sigma \in G} r_\sigma \sigma^*), \\ \psi_G^H: RG &\longrightarrow \bar{R}G^* \quad (\sum_{\sigma \in G} r_\sigma \sigma \longmapsto \sum_{\sigma \in G} \bar{r}_\sigma \sigma^*),\end{aligned}$$

where σ^* is the residue class of σ modulo H .

In what follows, we shall show that all the results in [3] are still valid without assuming that R is semi-primary. As to notations and terminologies used here without mention, we follow [3].

First, we shall prove a slight modification of [2; Lemma 2].

Lemma 1. *Let G be a finite p -group: $|G| = p^n$. If $J(R) = 0$ and $pR = 0$ then $J(RG) = \text{Ker } \phi_G$ and $(J(RG))^{p^n} = 0$.*

Proof. Evidently, $\text{Ker } \phi_G \supseteq J(RG)$. It remains therefore to prove $(\text{Ker } \phi_G)^{p^n} = 0$. We shall prove this by making use of the induction with respect to n . In case $n=1$, it is easy to see that $(\text{Ker } \phi_G)^p = (\sum_{\sigma \in G} R(1-\sigma))^p = 0$. Next, suppose $n > 1$ and that our assertion is true for $n-1$. Choose a normal subgroup H of G such that $(G:H) = p$, and set $G^* = G/H$. Then, by the case $n=1$, we have $(\phi_G^H(\text{Ker } \phi_G))^p = (\text{Ker } \phi_{G^*})^p = 0$, whence it follows $(\text{Ker } \phi_G)^p \subseteq \text{Ker } \phi_G^H = (\text{Ker } \phi_H)G$. Accordingly, by the induction hypothesis, $(\text{Ker } \phi_G)^{p^n} = ((\text{Ker } \phi_H)G)^{p^{n-1}} = (\text{Ker } \phi_H)^{p^{n-1}}G = 0$.

Now, by the light of Lemma 1, we can prove [3; Th. 2] without assuming that R is semi-primary.

Theorem 1. *Let G be a locally finite p -group. If $p\bar{R} = 0$ then $J(RG) = \text{Ker } \psi_G$.*

Proof. It is enough to prove that $\text{Ker } \psi_{r_n} \subseteq J(RG)$. Let $x = \sum_1^n r_i \sigma_i$ be an arbitrary element of $\text{Ker } \psi_{r_n}$, where $r_i \in R$ and $\sigma_i \in G$. We set $K = \langle \sigma_1, \dots, \sigma_n \rangle$. Then, K is a finite p -group and x is in $\text{Ker } \psi_K$. By Lemma 1, we have $\psi_K^{-1}(\text{Ker } \psi_K) \subseteq J(\overline{R}K)$. Hence, $(\text{Ker } \psi_K)^{|K|} \subseteq \text{Ker } \psi_K^{-1} = J(R)K \subseteq J(RK)$, whence it follows that x is quasi-regular.

Finally, we shall present the following which contains [3; Ths. 1 and 3] and [2; Th. 2].

Theorem 2. *Let $p\overline{R} = 0$. If H is a normal subgroup of G such that G/H is a locally finite p' -group then $J(RG) = J(RH)G$.*

Proof. As $J(RH)G \subseteq J(RG)$ by [3; Lemma 1], it remains only to prove the converse inclusion. Let $x = \sum_1^n a_i \sigma_i$ be an arbitrary element of $J(RG)$, where $a_i \in RH$ and $\sigma_i \in G$. Let $K = \langle H, \sigma_1, \dots, \sigma_n \rangle$. Then, it is evident that K/H is a finite p' -group. Now, patterning after the proof of [4; Prop. 1.5], one will easily see that $J(RK) = J(RH)K$. Combining this with [1; Prop. 9], we obtain

$$x \in J(RG) \cap RK \subseteq J(RK) = J(RH)K \subseteq J(RH)G,$$

completing the proof.

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