FURTHER RESULTS ON THE DISCRETE ANALYTIC DERIVATIVE EQUATION UNDER THE PRIME CONVOLUTION PRODUCT

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1. Introduction

The concept of a discrete analytic function was introduced by Jacqueline Ferrand [3], and many properties of discrete analytic functions were obtained by Duffin [4]. Duffin and Duris had studied a convolution product of the discrete analytic functions in [2]. Those functions are defined as follows. The points of the complex plane with integer coordinates are called lattice points. Consider a complex valued function defined on the complex plane. If f is defined on the square consisting of the lattice points $\{z,\ z+1,\ z+1+i,\ z+i\}$, then f is said to be discrete analytic on that square if $Lf \equiv f(z)+if(z+1)-f(z+1+i)-if(z+i)=0$. If $f(z)$ is discrete analytic on every square in a simply connected region R on the complex plane, $f(z)$ is said to be discrete analytic in R, and the set of all functions discrete analytic in R will be denoted by $A(R)$.

The line integral of $f(z)$, the double dot integral of $f(z)$ and $g(z)$, a convolution product for two functions $f(z)$ and $g(z)$ are defined respectively by

$$\int_{z_0}^{z_n} f(t)\delta t=\sum_{r=1}^{n} \frac{1}{2}[f(z_r)+f(z_{r-1})][z_r-z_{r-1}]$$

$$\int_{z_0}^{z_n} f(t):g(t)\delta t=\sum_{r=1}^{n} \frac{1}{4}[f(z_r)+f(z_{r-1})][g(z_r)+g(z_{r-1})](z_r-z_{r-1})$$

$$f*g(z)=\int_0^z f(z-t):g(t)\delta t$$

where R is a convolution region containing the origin and $f(z),\ g(z)\in A(R)$.

If $F\in A(R)$, the nth discrete derivative of F will be denoted by D^nF. In [2], it has been shown that there exists a unique discrete analytic function $F(z)$ such that

$$D^nF(z)+c_{n-1}D^{n-1}F(z)+\cdots\cdots+c_1DF(z)+c_0F(z)=f(z)$$

where $f(z)\in A(R)$, ± 2 and $\pm 2i$ are not roots of

$$r^n + c_{n-1}r^{n-1} + \cdots\cdots\cdots + c_1 r + c_0 = 0$$

and $F(z)$ is given by appropriate boundary conditions and c_i $(i=0, 1, 2, \cdots, n-1)$ is an arbitrary constant.

In [1], Duffin and Duris have discussed the general solution of a discrete derivative equation of the first order with constant coefficient. If $a^4 \neq 16$ then the general solution of $DF(z) - aF(z) = b(z)$ with $F(0) = c$ is

$$F(z) = ce(z, a) + \int_0^z e(z-t, a) : b(t)\delta t$$

where $b(z) \in A(R)$, c is an arbitrary constant and $e(z, a)$ is called the discrete exponential function which is defined as $e(z, a) = \left(\frac{2+a}{2-a}\right)^x \left(\frac{2+ia}{2-ia}\right)^y$ for $z = x + iy$. This function has been investigated previously by Ferrand [3] and Duffin [4].

In [5], the present author has shown that there exists a unique analytic function $F(z)$ in R such that $DF(z) - aK * F(z) = b(z)$ with $F(0) = c_0$ and $b(z) \in A(R)$, if $K(z) \in A(R)$ and $ah^2[K(0) + K(h)] \neq 8$ for $h = \pm 1$ or $\pm i$.

In [6], the author also has proved that if $ah^3[K(0) + K(h)] \neq 16$ for $h = \pm 1$ or $\pm i$, then there exists a unique function $F(z) \in A(R)$ such that $D^2F(z) - aK * F(z) = b(z)$ with $F(0) = c_0$ and $DF(0) = c_1$ where $K(z)$ and $b(z) \in A(R)$.

In [2], the prime convolution product of $f(z)$ and $g(z)$ is defined by

$$f *' g(z) = \int_0^z f(z-t) : g'(t)\delta t + f(z)g(0), \quad \text{i. e.} \quad f *' g = f * g' + f(z)g(0)$$

where the line integral is taken over a chain in R, whose counter chain is also in R and

$$\int_a^b f : g'(t)\delta t = \frac{1}{2}\sum_{r=1}^{n} [f(z_r) + f(z_{r-1})][g(z_r) - g(z_{r-1})], \quad (a = z_0, z_1, \cdots, z_n = b).$$

In [7], Deeter and Lord have shown that the ring $(A(R), +, *')$ can be embedded in a quotient structure. And they defined operators as elements of this quotient structure then developed an operational calculus on discrete analytic functions as Hayabara did in [8].

Section 2 of this paper is concerned with a solution of the discrete analytic derivative equation of the first order with a constant coefficient.

And the major purpose of this paper is to present the general solution of discrete analytic derivative equations of the first and the second order. Thus in sections 3 and 4, we will discuss the general solution of $DF(z)-aK*'F(z)=b(z)$ and $D^2F(z)-aK*'F(z)=b(z)$ respectively. In section 5, we briefly discuss the nth order. We only verify the solution for the points on the positive x-axis in the proof of Theorems 1, 3, and 5. This is sufficient because we can use a similar process to prove that is has a solution for the points on the negative x-axis and the whole y-axis. Then following the remarks of Duffin [4] a function $F(z)\in A(R)$ is uniquely determined by its values on the real and imaginary axes. Throughout this paper, R denotes a convolution region containing the origin.

2. First order nonhomogeneous equation

Theorem 1. *If* $a^4\neq 16$ *then there exists a unique solution* $F(z)$ *analytic in* R *such that*

(2.1) $$DF(z)-aF(z)=0 \text{ with } F(0)=c_0.$$

And $F(z)$ *has a form* $F(z)=c_0\left(\frac{2+a}{2-a}\right)^x\left(\frac{2+ia}{2-ia}\right)^y=c_0e(z,a)$ *where* $z=x+yi$.

Proof. Integrating both sides of (2.1), we have $F(z+h)-F(z)=a\int_z^{z+h}F(t)\delta t=\frac{1}{2}ah[F(z+h)+F(z)]$. Since $a^4\neq 16$, we obtain the stepping formula for the solution

(2.2) $$F(z+h)=\frac{2+ah}{2-ah}F(z) \quad \text{for } h=\pm 1 \text{ or } \pm i.$$

Then, in order to find a solution of (2.1), we integrate both sides of (2.1) from 0 to z, and then using the formulas [7] $l*'^n=\frac{z^{(n)}}{n!}$ and $l*'F(z)=\int_0^z F(t)\delta t$, we get

$$F(z)-F(0)=al*'F=az^{(1)}*'F=a\int_0^z(z-t)^{(1)}:F'(t)\delta t+aF(0)z^{(1)}, \text{ i. e.}$$

(2.3) $$F(z)=c_0(1+az^{(1)})+a\int_0^z(z-t)^{(1)}:F'(t)\delta t.$$

It remains to prove that (2.3) is a required solution. From (2.3) we get $F(h)=c_0(1+ah^{(1)})+\frac{1}{2}ah^{(1)}[F(h)-c_0]$. Since $2-ah^{(1)}\neq 0$ for $h=\pm 1$ or $\pm i$, we have $F(h)=\frac{2+ah^{(1)}}{2-ah^{(1)}}c_0$ and $F(1)=\frac{2+a}{2-a}c_0$. And then we get

$DF(1)=2[F(1)-F(0)]-DF(0)=aF(1)$, i. e. (2.1) has a solution for $z=1$. By induction, if (2.1) has a solution for $z=k$, then we can show (2.1) also has a solution for $z=k+1$. This follows from the following lemma.

Lemma 1. *If* $F(z)=c_0(1+az^{(1)})+a\int_0^z (z-t)^{(1)}:F'(t)\delta t$ *and* $DF(k)-aF(k)=0$ *then* $DF(k+1)-aF(k+1)=0$.

Proof. $\frac{1}{2}\{DF(k+1)+DF(k)\}=F(k+1)-F(k)$

$$=ac_0[(k+1)^{(1)}-k^{(1)}]+a\int_0^{k+1}(k+1-t)^{(1)}:F'(t)\delta t-a\int_0^k (k-t)^{(1)}:F'(t)\delta t$$
$$=ac_0+a\int_0^k[(k+1-t)^{(1)}-(k-t)^{(1)}]:F'(t)\delta t+a\int_k^{k+1}(k+1-t)^{(1)}:F'(t)\delta t$$
$$=\frac{1}{2}a\{F(k+1)+F(k)\}. \quad \text{q. e. d.}$$

Therefore, (2.1) has a solution for the points on the positive x-axis. Uniqueness of this solution $F(z)$ with $F(0)=c_0$ is clear from the stepping formula (2.2). Using (2.2), the general solution of (2.1) can be found, i. e.

$$F(z)=c_0\left(\frac{2+a}{2-a}\right)^x\left(\frac{2+ia}{2-ia}\right)^y=c_0e(z,a) \quad \text{for } z=x+iy.$$

Corollary. *If* $b(z)\in A(R)$ *and* $a^4\neq 16$ *then there exists a unique analytic function* $F(z)$ *in* R *such that* $DF(z)-aF(z)=b(z)$ *with* $F(0)=c_0$. *And the general solution* $F(z)$ *is given by*

$$F(z)=c_0(1+az^{(1)})+a\int_0^z (z-t)^{(1)}:F'(t)\delta t+\int_0^z b(t)\delta t$$

and the stepping formula is

$$F(z+h)=\frac{2+ah}{2-ah}F(z)+\frac{h[b(z+h)+b(z)]}{2-ah}.$$

The proof of this corollary is similar to that of Theorem 1.

3. The general case of the first order equations

Consider the general case of the first order homogeneous discrete analytic derivative equation

$$(3.1) \qquad DF(z)-aK*'F(z)=0 \text{ with } F(0)=c_0.$$

The following lemma is easy to prove.

Lemma 2. *If* $DF(z)-aK*'F(z)=0$ *with* $F(0)=c_0$ *then* $LF(z)=\frac{1}{2}iaL[K*'F(z)]$

Theorem 2. *Let* $K(z)\in A(R)$ *and if* (3.1) *has a solution in* R *and* $4-ah[K(0)+K(h)]\neq 0$ *for* $h=\pm 1$ *or* $\pm i$ *then this solution is discrete analytic in* R

Proof. $L[K*'F(z)]=\int_0^z LK(z-t):F'(t)\delta t+i\int_z^{z+1}K(z+1-t):F'(t)\delta t-\int_z^{z+1+i}K(z+1+i-t):F'(t)\delta t-i\int_z^{z+i}K(z+i-t):F'(t)\delta t+F(0)LK(z)$. Since $LK(z-t)=0$ and $LK(z)=0$, we have $L[K*'F(z)]=\frac{1}{2}[K(0)+K(i)]LF(z)$. By Lemma 2, we get $LF(z)=\frac{1}{4}ai[K(0)+K(i)]LF(z)$. From this it follows that $LF(z)=0$ since $4-ai[K(0)+K(i)]\neq 0$. Similarly, we also can obtain the following three expressions: $\{4-a[K(0)+K(1)]\}LF(z)=0$, $\{4+ai[K(0)+K(-i)]\}LF(z)=0$, and $\{4+a[K(0)+K(-1)]\}LF(z)=0$. Thus, if$4-ah[K(0)+K(h)]\neq 0$ for h equal to one of the values ± 1 or $\pm i$, the n$LF(z)=0$. This proves Theorem 2.

Theorem 3. *Let* $K(z)\in A(R)$ *and if* $4-ah[K(0)+K(h)]\neq 0$ *for* $h=\pm 1$ or $\pm i$ *then there exists a unique analytic function* $F(z)$ *in* R *such that* $DF(z)-aK*'F(z)=0$ *with* $F(0)=c_0$. *And the stepping formula of the solution is*

$$F(z+h)=F(z)+\frac{2ah}{4-ah[K(0)+K(h)]}\Big\{\int_0^z[K(z+h-t)+K(z-t)]:F'(t)\delta t+c_0[K(z+h)+K(z)]\Big\} \tag{3.2}$$

Proof. Integrating the expression $DF(z)=aK*'F(z)$ from z to $z+h$, and letting $G=K*'F$ we have $F(z+h)-F(z)=\frac{1}{2}ah[G(z+h)+G(z)]$ and $G(z+h)+G(z)=\int_z^{z+h}K(z+h-t):F'(t)\delta t+\int_0^z[K(z+h-t)+K(z-t)]:F'(t)\delta t+F(0)[K(z+h)+K(z)]=\frac{1}{2}[K(0)+K(h)][F(z+h)-F(z)]+\int_0^z[K(z+h-t)+K(z-t)]:F'(t)\delta t+F(0)[K(z+h)+K(z)]$, i. e. $F(z+h)-F(z)=\frac{1}{4}ah[K(0)+K(h)][F(z+h)-F(z)]+\frac{1}{2}ah\int_0^z[K(z+h-t)+K(z-t)]:F'(t)\delta t+\frac{1}{2}ahF(0)[K(z+h)+K(z)]$. By the hypothesis $4-ah[K(0)+K(h)]\neq 0$ we obtain the stepping formula. It remains to prove that the values which we get from (3.2) satisfy the equation (3.1). First, we will show that (3.1) has a solution for $z=1$. Since $F(1)=\frac{4+a\bar{K}(1)}{4-a\bar{K}(1)}c_0$ where $\bar{K}(1)=K(0)+K(1)$, we get $DF(1)=2[F(1)-c_0]-DF(0)$ and $aG(1)=aK*'F(1)=\frac{1}{2}a\bar{K}(1)F(1)+\frac{1}{2}ac_0[K(1)-K(0)]$. Since $DF(0)=aK(0)F(0)$ we have $DF(1)=ac_0\left[\frac{4K(1)}{4-a\bar{K}(1)}-K(0)\right]$. From this it is not difficult to show $DF(1)-aK*'F(1)=0$. By induction, we suppose that (3.1) is true for $z=m$, i. e. $DF(m)-aK*'F(m)=0$. We claim that $DF(m+1)-aK*'F(m+1)=0$. Before we show this, we first prove the following identity:

$$2\{F(m+1)-F(m)\}=aK*'F(m+1)+ak*'F(m) \quad \text{for } m\geq 0 \tag{3.3}$$

If $m=0$ or 1, it is true. Suppose it is true for $m=p$, i. e. $2\{F(p+1)-F(p)\}=aK*'F(p+1)+aK*'F(p)$. We claim that $2\{F(p+2)-F(p+1)\}=aK*'F(p+2)+aK*'F(p+1)$. It is sufficient to prove

$$2\{F(p+2)-F(p)\}=aK*'F(p+2)+2aK*'F(p+1)+aK*'F(p). \tag{3.4}$$

Since $aK*'F(p+2)+aK*'F(p+1)=\frac{1}{2}a\{\bar{K}(1)[F(p+2)-F(p+1)]+\bar{\bar{K}}(p+2)[F(1)-F(0)]+\bar{\bar{K}}(p+1)[F(2)-F(1)]+\cdots+\bar{\bar{K}}(2)[F(p+1)-F(p)]\}+a\bar{K}(p+2)F(0)$, (3.4) becomes

$$\begin{aligned} BF(p+2)=\;&a\bar{\bar{K}}(2)F(p+1)+F(p)\{B+a\bar{\bar{K}}(3)\}+aF(p-1)\\ &\{\bar{\bar{K}}(4)-\bar{\bar{K}}(2)\}+\cdots+aF(1)\{\bar{\bar{K}}(p+2)-\bar{\bar{K}}(p)\}+\\ &aF(0)\{\bar{K}(p+2)-\bar{K}(p)\} \end{aligned} \tag{3.5}$$

where $\bar{K}(p)=K(p)+K(p-1)$, $\bar{\bar{K}}(p)=\bar{K}(p)+\bar{K}(p-1)$ for $p\geq 1$, and $B=4-a\bar{K}(1)$.

On the other hand, by the stepping formula we have

$$\begin{aligned} BF(p+2)=\;&BF(p+1)+a\{\bar{\bar{K}}(p+2)[F(1)-F(0)]+\bar{\bar{K}}(p+1)\\ &[F(2)-F(1)]+\cdots+\bar{\bar{K}}(2)[F(p+1)-F(p)]\}\\ &+2aF(0)\bar{K}(p+2) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} BF(p+1)=\;&BF(p)+a\{\bar{\bar{K}}(p+1)[F(1)-F(0)]+\bar{\bar{K}}(p)\\ &[F(2)-F(1)]+\cdots+\bar{\bar{K}}(2)[F(p)-F(p-1)]\}\\ &+2aF(0)\bar{K}(p+1). \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6) and rearranging into polynomial form with respect to $F(p)$, we obtain (3.5). Therefore we have proved (3.4).

By the definition of the derivative $DF(m+1)+DF(m)=2[F(m+1)-F(m)]$ and using identity (3.3) we obtain $DF(m+1)+DF(m)=aK*'F(m+1)+aK*'F(m)$ for $m\geq 0$. Since $DF(m)=aK*'F(m)$, we have $DF(m+1)-aK*'F(m+1)=0$.

We have proved that (3.1) has a solution for the points on the positive x-axis. Uniqueness of this solution given by $F(0)=c_0$ is clear from the stepping formula and $F(z)\in A(R)$ is clear from Theorem 2. Similarly, we have the following corollary.

Corollary. *Let $K(z)$ and $b(z)$ be discrete analytic in R. If $4-ah[K(0)+K(h)]\neq 0$ for $h=\pm 1$ or $\pm i$, then there exists a unique function $F(z)$ discrete analytic in R such that*

$$DF(z)-aK*'F(z)=b(z) \quad \text{with} \quad F(0)=c_0. \tag{3.8}$$

And the solution of (3.8) can be calculated by the following stepping formula

$$F(z+h)=F(z)+\frac{2h}{4-ah[K(0)+K(h)]}\left\{a\int_0^z[K(z+h-t)+K(z-t)]:F'(t)\delta t\right.$$
$$\left.+ac_0[K(z+h)+K(z)]+b(z+h)+b(z)\right\}.$$

4. Discrete derivative equations of the second order

In this section we consider the case of a discrete nonhomogeneous derivative equation of the second order, i. e. $D^2F(z)-aK*'F(z)=b(z)$ with $DF(0)=c_1$ and $F(0)=c_0$.

In [7], Deeter and Lord have shown the following result.

Lemma 3. *Let $f(z)$ and $k(z)\in A(R)$. If $\lambda[k(0)+k(h)]\neq 2$ for $h=\pm 1$ or $\pm i$ then*

(I) *there exists a function $u(z)\in A(R)$ such that*

$$u(z)=f(z)+\lambda\int_0^z k(z-t):u'(t)\delta t, \tag{4.1}$$

(II) *the stepping formula is*

$$u(z+h)=u(z)+\frac{2}{2-\lambda[k(0)+k(h)]}\left\{f(z+h)-f(z)+\right.$$
$$\left.\lambda\int_0^z[k(z+h-t)-k(z-t)]:u'(t)\delta t\right\}. \tag{4.2}$$

Proof. Conclusion (I) is easily obtained by a method similar to the proof of Theorem 14 in [7]. From (4.1) we have

$$u(z+h)-u(z)=f(z+h)-f(z)+\lambda\int_0^z[k(z+h-t)-k(z-t)]:u'(t)\delta t$$
$$+\lambda\int_z^{z+h}k(z+h-t):u'(t)\delta t$$

and then we obtain

$$\left\{2-\lambda[k(0)+k(h)]\right\}\left\{u(z+h)-u(z)\right\}=2[f(z+h)-f(z)]$$
$$+2\lambda\int_0^z[k(z+h-t)-k(z-t)]:u'(t)\delta t.$$

This completes the proof of conclusion (II).

In [8], Hayabara has shown the following:

Lemma 4. *For* $G\in A(R)$, $n!\int_0^z\int_0^{t_1}\cdots\int_0^{t_n}G(t_{n+1})\delta t_{n+1}\cdots\delta t_1=\int_0^z(z-t)^{(n)}:G(t)\delta t$ *where* $z^{(n+1)}=(n+1)\int_0^z t^{(n)}\delta t$, $z^{(0)}=1$.

In [2], it is shown that the convolution product and the prime convolution product for f and $g\in A(R)$ respectively are commutative, associative and distributive over usual pointwise addition. Using the commutativity of the convolution product and the definition of the prime convolution product we have $f*(g*'h)=[g*h'+g(z)h(0)]*f$. And then by the distributivity $f*(g*'h)=(g*h')*f+g(z)h(0)*f(z)$. On the other hand $(f*g)*'h=(f*g)*h'+(f*g)(z)h(0)$. Using the associativity and commutativity of the convolution product we obtain the following result.

Lemma 5. $f*(g*'h)=(f*g)*'h$ *for* $f, g, h\in A(R)$.

Theorem 4. *Let* $K(z)\in A(R)$. *If* $8-ah^2[K(0)+K(h)]=0$ *for* $h=\pm 1$ *or* $\pm i$, *Then*

$$D^2F(z)-aK*'F(z)=0 \tag{4.3}$$

with $DF(0)=c_1$ *and* $F(0)=c_0$ *has no solution for* $z=h$ *if* $2c_0+hc_1\neq 0$.

Proof. Suppose, there exists a solution of (4.3) for $z=h$, integrating (4.3) from 0 to h, we get $DF(h)-DF(0)=a\int_0^h G(t)\delta t$ where $G=K*'F$. And then we have $DF(h)-DF(0)=\frac{1}{2}ah[G(h)+G(0)]=\frac{1}{4}ah[K(0)+K(h)][F(h)+c_0]$.

Using the definition of the derivative we have $h\{DF(h)+DF(0)\}=2[F(h)-F(0)]$. Therefore we get $\{8-ah^2[K(0)+K(h)]\}F(h)=\{8+ah^2[K(0)+K(h)]\}c_0+8hc_1$.

By the hypothesis if $8-ah^2[K(0)+K(h)]=0$ then $2c_0+hc_1=0$. This contradicts the assumption. q. e. d.

In order to find the stepping formula of the solution of (4.3), we integrate twice from 0 to z: $F(z)=a\int_0^z\int_0^{t_1}G(t)\delta t\delta t_1+c_1z+c_0$. Using Lemma 4 and Lemma 5, this becomes (4.1), i. e. $F(z)=az*K*'F(z)+c_1z+c_0$.

Let $H(z)=z*K(z)$. Then

$$F(z)=a\int_0^z H(z-t):F'(t)\delta t+aH(z)F(0)+c_1z+c_0. \tag{4.4}$$

Assuming $a[H(0)+H(h)]\neq 2$ and $K(z)\in A(R)$, and then using Lemma 3 we obtain that (4.4) has a solution $F(z)\in A(R)$ and the solution $F(z)$ can be calculated by

$$F(z+h)=F(z)+\frac{1}{2-a[H(0)+H(h)]}\Big\{2[E(z+h)-E(z)]+ 2a\int_0^z[H(z+h-t)-H(z-t)]:F'(t)\delta t\Big\}$$

where $E(z)=aH(z)c_0+c_1z+c_0$, $H(0)=0$, and

$$H(z)=z*K(z)=\sum_{n=1}^{m}\frac{1}{4}(2z-t_n-t_{n-1})[K(t_n)+K(t_{n-1})](t_n-t_{n-1}).$$

Since $E(z+h)-E(z)=ac_0[H(z+h)-H(z)]+c_1h$ and the condition $a[H(0)+H(h)]\neq 2$ is equivalent to the condition $8-ah^2[K(0)+K(h)]\neq 0$, the required stepping formula can be written in the form

$$F(z+h)=F(z)+\frac{8}{8-ah^2[K(0)+K(h)]}\Big\{ac_0[H(z+h)-H(z)]+ c_1h+a\int_0^z[H(z+h-t)-H(z-t)]:F'(t)\delta t\Big\} \tag{4.5}$$

where $H(z)=z*K(z)$.

Lemma 6. *For* $n\geq 2$

$$4\Big\{F(n)-F(n-1)+2\sum_{j=1}^{n-1}(-1)^{j+n}[F(j)-F(j-1)]+(-1)^n c_1\Big\} =aK*'F(n)+aK*'F(n-1). \tag{4.6}$$

Proof. It is clear for $n=2$. Suppose (4.6) is true for $n=p$, i. e.

$$4\Big\{F(p)-F(p-1)+2\sum_{j=1}^{p-1}(-1)^{j+p}[F(j)-F(j-1)]+(-1)^p c_1\Big\} =aK*'F(p)+aK*'F(p-1).$$

We want to show

$$4\left\{F(p+1)-F(p)+2\sum_{j=1}^{p}(-1)^{j+1+p}[F(j)-F(j-1)]+(-1)^{p+1}c_1\right\}$$
$$=aK*'F(p+1)+aK*'F(p).$$

Adding both expressions, it is sufficient to prove

$$(4.7)\qquad 4\{F(p+1)-2F(p)+F(p-1)\}=aK*'F(p+1)+2aK*'F(p)+aK*'F(p-1).$$

However, $aK*'F(p+1)+aK*'F(p)=\frac{1}{2}a\Big\{\overline{K}(1)[F(p+1)-F(p)]$ $+\sum_{j=0}^{p-1}\overline{\overline{K}}(p+1-j)[F(j+1)-F(j)]+2\overline{K}(p+1)F(0)\Big\}$. Hence, rearranging the right-hand side of (4.7) into the polynomial in terms of $F(j)$, $j=0, 1, 2, \cdots, p+1$, it becomes

$$aK*'F(p+1)+2aK*'F(p)+aK*'F(p-1)=$$
$$\frac{1}{2}a\Big\{F(0)[\overline{\overline{K}}(p+1)-\overline{\overline{K}}(p)]+\sum_{j=1}^{p-2}F(j)[\overline{\overline{K}}(p+2-j)-\overline{\overline{K}}(p-j)]$$
$$+F(p-1)[\overline{\overline{K}}(3)-\overline{K}(1)]+F(p)\overline{\overline{K}}(2)+F(p+1)\overline{K}(1)\Big\}.$$

Letting $B=8-a\overline{K}(1)$ and substituting into (4.7) we have

$$(4.8)\qquad BF(p+1)=aF(0)[\overline{\overline{K}}(p+1)-\overline{\overline{K}}(p)]+aF(1)[\overline{\overline{K}}(p+1)-\overline{\overline{K}}(p-1)]$$
$$+\cdots+F(p-1)[a\overline{\overline{K}}(3)-a\overline{K}(1)-8]+F(p)[a\overline{\overline{K}}(2)+16].$$

By the stepping formula we have

$$(4.9)\qquad BF(p+1)=BF(p)+ac_0\overline{\overline{K}}(p+1)+aF(1)[\overline{\overline{K}}(p+1)+\overline{\overline{K}}(p)]+\cdots$$
$$+aF(p-1)[\overline{\overline{K}}(3)+\overline{\overline{K}}(2)]+aF(p)[\overline{\overline{K}}(2)+2\overline{K}(1)]+8c_1.$$

Similarly,

$$(4.10)\qquad BF(p)=BF(p-1)+ac_0\overline{\overline{K}}(p)+aF(1)[\overline{\overline{K}}(p)+\overline{\overline{K}}(p-1)]+\cdots$$
$$+aF(p-2)[\overline{\overline{K}}(3)+\overline{\overline{K}}(2)]+aF(p-1)[\overline{\overline{K}}(2)+2\overline{K}(1)]+8c_1.$$

Subtracting (4.9) from (4.10) we obtain

$$(4.11)\qquad BF(p+1)=2BF(p)-BF(p-1)+ac_0[\overline{\overline{K}}(p+1)-\overline{\overline{K}}(p)]$$
$$+aF(1)[\overline{\overline{K}}(p+1)-\overline{\overline{K}}(p-1)]+\cdots$$
$$+aF(p-1)[\overline{\overline{K}}(3)-2\overline{K}(1)]+aF(p)[\overline{\overline{K}}(2)+2\overline{K}(1)].$$

Since (4.8) = (4.11), this lemma is proved.

Theorem 5. *Let* $K(z)\in A(R)$. *If* $8-ah^2[K(0)+K(h)]\neq 0$ *then*

$D^2F(z)-aK*'F(z)=0$ with $DF(0)=c_1$ and $F(0)=c_0$ has a solution $F(z) \in A(R)$ and the solution can be calculated by the stepping formula (4.5).

Proof. (4.3) has a solution for $z=1$, since we can calculate as follows: $D^2F(1)=4\{F(1)-F(0)-DF(0)\}-D^2F(0)$, $D^2F(0)=aK*'F(0)=ac_0K(0)$, and $D^2F(1)=4\{F(1)-c_0-c_1\}-ac_0K(0)$.

From (4.5) $F(1)=F(0)+\frac{8}{B}\{ac_0[H(1)-H(0)]+c_1\}$. Since $H(1)=\frac{1}{4}\overline{K}(1)$ and $H(0)=0$ we get

$$(4.12)\qquad \begin{aligned} &BF(1)=8c_0+ac_0\overline{K}(1)+8c_1 \\ &\text{and } D^2F(1)=\frac{4a\overline{K}(1)}{B}\{2c_0+c_1\}-ac_0K(0). \end{aligned}$$

On the other hand

$$(4.13)\qquad aK*'F(1)=\frac{a\overline{K}(1)}{B}\{ac_0\overline{K}(1)+4c_1\}+aK(1)c_0.$$

Since the right hand side of (4.12) equals the right hand side of (4.13), we obtain $D^2F(1)-aK*'F(1)=0$.

By induction, suppose $D^2F(m-1)-aK*'F(m-1)=0$. Then using Lemma 6 we have

$$\begin{aligned} D^2F(m)&=4\{F(m)-F(m-1)-DF(m-1)\}-aK*'F(m-1) \\ &=4\left\{F(m)-F(m-1)+2\sum_{j=1}^{m-1}(-1)^{m+j}[F(j)-F(j-1)]+(-1)^m c_1\right\} \\ &\quad -aK*'F(m-1)=aK*'F(m). \end{aligned}$$

q. e. d.

Corollary. *Let $K(z)$ and $b(z)$ be discrete analytic in R. If $8-ah^2[K(0)+K(h)]\neq 0$ for h equal to one of the values ± 1 or $\pm i$, then there exists a solution $F(z)$ discrete analytic in R such that*

$$(4.14)\qquad D^2F(z)-aK*'F(z)=b(z)$$

with $DF(0)=c_1$ and $F(0)=c_0$. And the solution of (4.14) *can be calculated by the following stepping formula*

$$(4.15)\qquad \begin{aligned} F(z+h)=F(z)+&\frac{8}{8-ah^2[K(0)+K(h)]}\Big\{ac_0[H(z+h)-H(z)] \\ &+M(z+h)-M(z)+c_1h+a\int_0^z[H(z+h-t)-H(z-t)]:F'(t)\delta t\Big\} \end{aligned}$$

with $D^2F(0)=ac_0K(0)+b(0)$ where $H=z*K$ and $M=z*b$.

With a proof similar to the proof of Theorem 5, we see that the function $F(z)$ which is obtained from (4.15) is exactly a solution of (4.14) and

$F(z) \in A(R)$.

5. Discrete derivative equations of the nth order

In this section, we briefly discuss the solution of the discrete derivative equations of the nth order. Integrating $D^n F(z) - aK*'F(z) = 0$ with $D^j F(0) = c_j$ $(j = 0, 1, 2, \cdots, n-1)$ from 0 to h, we have

$$(5.1) \qquad D^{n-1}F(h) - D^{n-1}F(0) = \frac{1}{4} ah[K(0) + K(h)][F(h) + c_0].$$

Using the definition of the derivative, we get

$$(5.2) \qquad D^{n-1}F(h) + c_{n-1} = \frac{2}{h}[D^{n-2}F(h) - c_{n-2}].$$

From (5.1) and (5.2) we have

$$D^{n-2}F(h) = \frac{ah^2}{2^3}[K(0) + K(h)][F(h) + c_0] + hc_{n-1} + c_{n-2}.$$

Using the definition again, we get

$$D^{n-j}F(h) = \frac{ah^j}{2^{j+1}}[K(0) + K(h)][F(h) + c_0] + \frac{h^{j-1}}{2^{j-2}} c_{n-1} + \cdots + hc_{n-j+1} + c_{n-j}.$$

Therefore we have $F(h) = \frac{ah^n}{2^{n+1}}[K(0) + K(h)]F(h) + \frac{ah^n}{2^{n+1}}[K(0) + K(h)]$ $c_0 + \frac{h^{n-1}}{2^{n-2}} c_{n-1} + \cdots + hc_1 + c_0$, i. e. $\{2^{n+1} - ah^n[K(0) + K(h)]\}F(h) = \{2^{n+1} + ah^n$ $[K(0) + K(h)]\}c_0 + 4\sum_{j=1}^{n-1} 2^{n-j} h^j c_j$. If $2^{n+1} - ah^n[K(0) + K(h)] = 0$ then $\sum_{j=0}^{n-1} 2^{n-j}$ $h^j c_j = 0$. We obtain the following result.

Theorem 6. *Let* $K(z) \in A(R)$. *If* $2^{n+1} - ah^n[K(0) + K(h)] = 0$ *for* $h = \pm 1$ *or* $\pm i$ *then* $D^n F(z) - aK*'F(z) = 0$ *with* $D^j F(0) = c_j$, $j = 0, 1, \cdots n-1$, *has no solution for* $z = h$ *if* $\sum_{j=0}^{n-1} 2^{n-j} h^j c_j \neq 0$.

In conclusion, if $K(z)$ and $b(z)$ are analytic in R and $2^{n+1} - ah^n$ $[K(0) + K(h)] \neq 0$ for $h = \pm 1$ or $\pm i$, by the use of the process in Sections 3 and 4, we can find the discrete analytic solution $F(z)$ in R such that $D^n F(z) - aK*'F(z) = b(z)$ with $D^j F(0) = c_j$, $j = 0, 1, 2, \cdots, n-1$, where $K(z)$, $b(z) \in A(R)$. And the stepping formula for the solution also can be found. However, the proof is more complicate if the order is higher.

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