ON A GROUP OF CYCLIC EXTENSIONS OVER COMMUTATIVE RINGS

Dedicated to Professor MASARU OSIMA on his 60th birthday

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Let R be a commutative ring with identity and let H be a commutative Hopf R-algebra with antipode [3] which is a flat R-module. In [3, chap. I], it is shown that the collection of isomorphism classes of Galois H-objects $E(H)$ in the category of commutative R-algebras forms an abelian group. If H is the dual Hopf R-algebra of a group algebra RG, where G is a finite group, then an arbitrary Galois H-object is a Galois extension of R with Galois group G [3, p. 59] in the sense of Chase-Harrison-Rosenberg [2]. In this paper we compute $E(H)$ for a group algebra which satisfies some conditions. In §1 we describe how to construct an abelian group $E(H)$ of a finite commutative Hopf R-algebra H which is given in [3]. In §2 we generalize the well known description of normal separable extensions of degree p of a field of characteristic p, and in §3 we give some similar results of §2. The materials of §2 and §3 are closely related to recent papers [8] and [9].

Throughout the following discussion R will be a fixed commutative ring with an identity 1 and all modules will be unital. Moreover, every ring has an identity which is preserved by every homomorphism, and all ring extensions will be assumed to have identities coinciding with the identity of the base ring. Unadorned $\otimes$ will mean $\otimes_R$.

1. Preliminaries.

Let H be a Hopf algebra with algebra structure maps $\mu : H\otimes H \longrightarrow H$, $\eta : R \longrightarrow H$, and coalgebra structure maps $\Delta : H \longrightarrow H\otimes H$, $\varepsilon : H \longrightarrow R$. An *H-object* is a pair (A, α), where A is a commutative R-algebra and $\alpha : A \longrightarrow A\otimes H$ is an R-algebra homomorphism such that

$$(\alpha\otimes 1)\alpha = (1\otimes\Delta)\alpha : A \longrightarrow A\otimes H\otimes H$$

and

$$(1\otimes\varepsilon)\alpha = 1_A : A \longrightarrow A\otimes R = A.$$

For brevity, we shall usually denote the pair (A, α) by the symbol A alone. When the map α needs explicit mention, we shall write $\alpha = \alpha_A$.

A homomorphism $f: A \longrightarrow B$ of H-objects is an R-algebra homomorphism such that $(f\otimes 1_H)\alpha_A=\alpha_B f$. Now let (A, α_A) be an H-object. We define the R-algebra homomorphism $\gamma_A: A\otimes A \longrightarrow A\otimes H$ by the formula

$$\gamma_A(x\otimes y)=(x\otimes 1)\alpha_A(y).$$

A will be called a *Galois H-object* if the following conditions hold:

(a) A is a faithfully flat R-module.

(b) $\gamma_A: A\otimes A \longrightarrow A\otimes H$ is an isomorphism.

Let H be a commutative Hopf algebra, A, B, C Galois H-objects, and $B\cong C$ as H-objects. Then $A\otimes B$ is a Galois $H\otimes H$-object with the structure map

$$A\otimes B \xrightarrow{\alpha_A\otimes\alpha_B} A\otimes H\otimes B\otimes H \xrightarrow{1\otimes\iota\otimes 1} A\otimes B\otimes H\otimes H$$

and $A\otimes B\cong A\otimes C$ as $H\otimes H$-objects. Furthermore,

$$(1) \qquad \begin{tikzcd} \widetilde{\Delta}(A\otimes B) \arrow[r] & A\otimes B\otimes H \arrow[r, shift left, "\omega"] \arrow[r, shift right, "\delta"'] & A\otimes B\otimes H\otimes H\otimes H \end{tikzcd}$$

is an equalizer diagram in the category of commutative R-algebras, with $\widetilde{\Delta}(A\otimes B)=\{x\in A\otimes B\otimes H\,;\,\omega(x)=\delta(x)\}$ [3, p. 31], where ω and δ are the compositions

$$\omega: A\otimes B\otimes H \xrightarrow{1\otimes 1\otimes \Delta} A\otimes B\otimes H\otimes H \xrightarrow{1\otimes 1\otimes \Delta\otimes 1} A\otimes B\otimes H\otimes H\otimes H,$$

and

$$\delta: A\otimes B\otimes H \xrightarrow{\alpha_A\otimes\alpha_B\otimes 1} A\otimes H\otimes B\otimes H\otimes H \xrightarrow{1\otimes\iota\otimes 1\otimes 1} A\otimes B\otimes H\otimes H\otimes H$$

respectively $(t: H\otimes B\ni h\otimes b \longmapsto b\otimes h\in B\otimes H)$, and the unlabeled map is the inclusion.

Theorem 1.1. ([3, Th. 2. 20]). *$\widetilde{\Delta}(A\otimes B)$ is a Galois H-object with the structure map $\alpha: \widetilde{\Delta}(A\otimes B) \longrightarrow \widetilde{\Delta}(A\otimes B)\otimes H$ which is induced by the equalizer diagram* (1).

If H is a commutative Hopf algebra with antipode which is a finitely generated projective R-module, we shall denote by $E(H)$ the set of H-isomorphism classes of Galois H-objects. Then $E(H)$ is an abelian group, with addition

$$(A)+(B)=(\widetilde{\Delta}(A\otimes B)) \quad ((A), (B) \text{ in } E(H))$$

and (H) is the zero element of $E(H)$ [3, Th. 3.9].

Now let G be a finite group. For the group algebra RG of G over R, we write $GR=\mathrm{Hom}_R(RG, R)$, which is a Hopf algebra with the usual structure maps (in the sense of [3, p. 59]).

Theorem 1.2 ([3, p. 59]). *Let A be a commutative R-algebra. Then A is a Galois GR-object if and only if it is a Galois extension of R with Galois group G in the sense of* [2].

Remark 1.3. A G-Galois extension (resp. a Galois GR-object) will be called a Galois GR-object (resp. a G-Galois extension) by the correspondence G-Galois extensions and Galois GR-objects as in the proof of Th. 1.2.

Let A, B be G-Galois extensions of R. Then

(1) A is isomorphic to B as GR-object if and only if there exists an R-algebra isomorphism $\varphi: A \longrightarrow B$ and a group automorphism ρ of G such that $\varphi(\sigma x)=\rho(\sigma)\varphi(x)$ $(\sigma\in G, x\in A)$.

(2) When A, B has no proper idempotents, A is isomorphic to B as GR-object if and only if A is isomorphic to B as R-algebra (cf. [5]).

Remark 1.4. Let $G=(\sigma)$ be a cyclic group of order n. Then $GR=\sum_{i=0}^{n-1}\oplus Rv_i$ $(v_i(\sigma^j)=1$ if $i=j$; $v_i(\sigma^j)=0$ if $i\neq j)$ is a finitely generated projective R-module which is a commutative Hopf algebra with

algebra structure: $\mu(v_i\otimes v_j)=v_iv_j=\begin{cases} v_i, & \text{if } i=j \\ 0, & \text{if } i\neq j, \end{cases}$

$\eta(r)=r$ $(r$ in $R)$,

coalgebra structure: $\Delta(v_i)=\sum_{j=0}^{n-1}v_j\otimes v_{\overline{i-j}}$, where $\overline{i-j}=i-j \pmod{n}$,

$\varepsilon(v_i)=\begin{cases} 1, & \text{if } i=0 \\ 0, & \text{if } i\neq 0, \end{cases}$

antipode: $\lambda(v_i)=v_{n-i}$.

2. A group of cyclic p-extensions.

Throughout this section R will be a commutative algebra over the prime field $GF(p)$ $(p\neq 0)$, and G will be a finite cyclic group of order p. First, we shall prove the following

Lemma 2.1. *Let r be an arbitrary element in R. Then $[X, r]=R[X]/(X^p-X-r)$ is a Galois GR-object with the structure map $\alpha: [X, r]$*

$\longrightarrow [X, r]\otimes GR$ which is defined by $\alpha(\bar{X})=\sum_{i=0}^{p-1}(\bar{X}+i)\otimes v_i$, where $\bar{X}=X+(X^p-X-r)$. Moreover, if A is an arbitrary Galois GR-object, then A is isomorphic to $R[X]/(X^p-X-r)$ as Galois GR-object for some r in R. Therefore we may write $A=[X, r]$.

Proof. By [8, Th. 1.1], $[X, r]$ is a Galois extension of R with a Galois group generated by an automorphism $\sigma: \bar{X}\longrightarrow \bar{X}+1$. Then we have $\alpha(a)=\sum_{i=0}^{p-1}\sigma^i(a)\otimes v_i$ for a in $[X, r]$. Hence, by the discussion on [3, p. 59], we see that $[X, r]$ is a Galois GR-object with the structure map α. Moreover, if A is a G-Galois extension of R then by [8, Th. 1.2], A is isomorphic to $[X, r]$ for some r in R as G-Galois extension, and so as Galois GR-object.

The following theorem is useful in this section.

Theorem 2.2. *Let $A=[X, r]$ and $B=[Y, s]$ be Galois GR-objects. Then $\widetilde{J}(A\otimes B)=[Z, r+s]$.*

Proof. If v_i is defined as in Remark 1.4, we evidently have

$$\textstyle\sum_{i=0}^{p-1} i(v_j\otimes v_{\overline{i-j}})=\sum_{i=0}^{p-1}(i+j)(v_j\otimes v_i)$$

where $0\leqq j\leqq p-1$, and

$$\textstyle\sum_{i,j=0}^{p-1}(i+j)(v_i\otimes v_j)=\sum_{i=0}^{p-1} i(v_i\otimes 1+1\otimes v_i).$$

We set $z=\bar{X}\otimes 1\otimes 1+1\otimes\bar{Y}\otimes 1+\sum_{i=0}^{p-1} i(1\otimes 1\otimes v_i)$. Then by the preceding equalities, we have

$$\begin{aligned}(\omega-\delta)(z)&=(\omega-\delta)(\bar{X}\otimes 1\otimes 1+1\otimes\bar{Y}\otimes 1)+(\omega-\delta)\left(\textstyle\sum_{i=0}^{p-1} i(1\otimes 1\otimes v_i)\right)\\ &=-1\otimes 1\otimes\textstyle\sum_{i=0}^{p-1} i(v_i\otimes 1+1\otimes v_i)\otimes 1\\ &\qquad+1\otimes 1\otimes\textstyle\sum_{i,j,k=0}^{p-1} i(v_k\otimes v_{\overline{j-k}}\otimes v_{\overline{i-j}})\\ &\qquad-1\otimes 1\otimes\textstyle\sum_{i=0}^{p-1} i(1\otimes 1\otimes v_i)\\ &=-1\otimes 1\otimes\textstyle\sum_{i,j=0}^{p-1}(i+j)(v_i\otimes v_j)\otimes 1\\ &\qquad+1\otimes 1\otimes\textstyle\sum_{i,j,k=0}^{p-1}(i+j+k)(v_k\otimes v_j\otimes v_i)\\ &\qquad-1\otimes 1\otimes\textstyle\sum_{i=0}^{p-1} i(1\otimes 1\otimes v_i).\end{aligned}$$

Since $\sum_{i=0}^{p-1} v_i=1$, we have $\omega(z)=\delta(z)$, that is, z is in $\widetilde{J}(A\otimes B)$. Moreover, noting that the v_i are orthogonal idempotents, we have $z^p-z=r+s$.

Let $\alpha^*: \widetilde{J}(A\otimes B)\longrightarrow\widetilde{J}(A\otimes B)\otimes GR$ be the structure map which is induced by (1). Then α^* is the unique homomorphism such that the diagram below is commutative

$$\begin{tikzcd}
\tilde{\Delta}(A\otimes B) \arrow[r] \arrow[d, "\alpha^*"'] & A\otimes B\otimes GR \arrow[d, "1\otimes 1\otimes \Delta"] \\
\tilde{\Delta}(A\otimes B)\otimes GR \arrow[r] & A\otimes B\otimes GR\otimes GR
\end{tikzcd}$$

where the unlabeled arrows denote the inclusions. We consider here the following diagram

$$\begin{tikzcd}
{[Z, r+s]} \arrow[r, "f"] \arrow[d, "\alpha"'] & \tilde{\Delta}(A\otimes B) \arrow[d, "\alpha^*"] \\
{[Z, r+s]\otimes GR} \arrow[r, "f\otimes 1"] & \tilde{\Delta}(A\otimes B)\otimes GR
\end{tikzcd}$$

where f is an R-algebra homomorphism mapping $\bar{Z}$ into z. Then we have

$$\begin{aligned}
&\alpha^*(z)-\textstyle\sum_{i=0}^{p-1}(z+i)\otimes v_i\\
&=(1\otimes 1\otimes \Delta)(z)-(z\otimes 1+\textstyle\sum_{i=0}^{p-1}i(1\otimes 1\otimes 1\otimes v_i))\\
&=\textstyle\sum_{i,j=0}^{p-1}(i+j)(1\otimes 1\otimes v_j\otimes v_i)\\
&\quad-\textstyle\sum_{i=0}^{p-1}i(1\otimes 1\otimes 1\otimes v_i+1\otimes 1\otimes v_i\otimes 1)\\
&=0
\end{aligned}$$

in $A\otimes B\otimes GR\otimes GR$. That is, $\alpha^*(z)=\sum_{i=0}^{p-1}(z+i)\otimes v_i$ and so $(f\otimes 1)\alpha=\alpha^* f$. Thus f is (homomorphic, and hence by [3, Th. 1.12]) isomorphic to $\tilde{\Delta}(A\otimes B)$ as GR-object. Hence $\tilde{\Delta}(A\otimes B)=[Z, r+s]$.

Corollary 2.3. *$[X, s]\cong GR$ if and only if $s=r_0^p-r_0$ for some r_0 in R.*

Proof. Since $([X, 0])+([X, r])=([X, r])$ for all $([X, r])$ in $E(GR)$, we have $([X, 0])=(GR)$, that is, $[X, 0]\cong GR$. Noting that $\{1, \bar{X}, \cdots, \bar{X}^{p-1}\}$ is a basis of $[X, 0]$, it follows that $a^p-a\in\{r^p-r\,;\, r \text{ in } R\}$ for every $a\in[X, 0]$. Hence, if $[X, s]\cong[X, 0]$ then $s=\bar{X}^p-\bar{X}\in\{r^p-r\,;\, r \text{ in } R\}$.

To see the converse, let $s=r_0^p-r_0$ for some r_0 in R, $y=\bar{X}-r_0\in[X, s]$, and $z=\bar{X}\in[X, 0]$. Then we have $R[y]\cong R[z]$ (as GR-object), mapping y into z. Thus we obtain $[X, s]\cong[X, 0]$.

In virture of the preceding corollary, we obtain

Theorem 2.4. *$E(GR)\cong R^+/\{r^p-r\,;\, r \text{ in } R\}$ as groups, where R^+ is the additive group of R.*

Let R be a ring without proper idempotents, Ω a separable closure of R in the sense of G. J. Janusz [5, Def. 3], and Π the set of R-algebra automorphisms of Ω. Moreover, we denote by $\mathcal{F}$ the set of subrings of Ω which are G-Galois over R. Then we obtain the following lemma by [8, Th. 1.6], Cor. 2.3, Remark 1.1 (2) and the results of [5], [7].

Lemma 2.5. *Let R be a ring without proper idempotents, and A a G-Galois extension of R. Then*

(1) *$A \not\cong GR$ as GR-objects if and only if A has no proper idempotents, and which is equivalent to that $A \cong S$ as R-algebras for some $S \in \mathcal{F}$.*

(2) *For S_1, $S_2 \in \mathcal{F}$, $S_1 \cong S_2$ as GR-objects if and only if $S_1 = S_2$.*

Now let $C \in E(GR)$. If $C \not\cong (GR)$ then, by Lemma 2.5, there exists a unique element S in $\mathcal{F}$ with $(S) = C$, and we write $C' = S$. Moreover, if $C = (GR)$, we write $C' = R$. Then by Lemma 2.5, we have a one-to-one correspondence $E(GR) \longrightarrow \{\mathcal{F}, R\}$ mapping C into C'. In this situation, we have

Lemma 2.6. *Let $G = \{0, 1, \cdots, p-1\} \subset R$, and $\varphi \in \mathrm{Hom}_c(\Pi, G)$. Then there exists an element a in Ω such that $\sigma(a) = a + \varphi(\sigma)$ for every $\sigma \in \Pi$. In this case, there holds $\Omega^{\mathrm{Ker}(\varphi)} = ([X, a^p - a])'$.*

Proof. If $\varphi = 0$ then, for every $\sigma \in \Pi$, $\varphi(\sigma) = 0$, where $\sigma(a) = a + \varphi(\sigma)$ for any $a \in R$. Let $\varphi \neq 0$, and $A = \Omega^{\mathrm{Ker}(\varphi)}$. Then there exists an element $\tau \in \Pi$ with $\varphi(\tau) = 1$. Moreover, we have $A = ([X, r])'$ for some $([X, r]) \in E(GR)$, and so $A = R[c]$, $c^p - c = r$ for some c. Since $\tau \notin \mathrm{Ker}(\varphi)$, we have $\tau(c) = c + j$, $1 \leqq j \leqq p-1$. Set $a = j^{-1}c$. Then $\tau(a) = a + 1 = a + \varphi(\tau)$. Noting $\Pi = \cup_{i=0}^{p-1} \tau^i \mathrm{Ker}(\varphi)$, it follows that $\sigma(a) = a + \varphi(\sigma)$ for every $\sigma \in \Pi$. This implies that $\sigma \in \mathrm{Ker}(\varphi)$ if and only if $\sigma(a) = a$. Since $a^p - a \in R$, $R[a]$ is separable over R by [8, Lemma 1.1]. Hence, by [7] and [8, Lemma 1.1], we obtain $\Omega^{\mathrm{Ker}(\varphi)} = R[a] = ([X, a^p - a])'$.

Now we shall prove the following theorem which corresponds to the result of D. K. Harrison [4, Th. 4] and S. U. Chase [1, Th. 3. 5].

Theorem 2.7. *If R has no proper idempotents, then*

$$\mathrm{Hom}_c(\Pi, G) \cong E(GR)$$

with Π the group of automorphism of a separable closure Ω of R in the sense of Janusz [5, Def. 3], *and the left-hand side denoting continuous homomorphisms from the compact group Π to the discrete group G.*

Proof. We consider a correspondence h: $\mathrm{Hom}_c(\Pi, G) \longrightarrow E(GR)$ defined by

$$\varphi \longmapsto \Omega^{\mathrm{Ker}(\varphi)} = C' \longmapsto C = h(\varphi) \quad (\varphi \in \mathrm{Hom}_c(\Pi, G)).$$

Clearly h is a mapping which is bijective. Let $G = \{0, 1, \cdots, p-1\} \subset R$ and $\varphi, \psi \in \mathrm{Hom}_c(\Pi, G)$. Then by Lemma 2.6, there exist elements $a, b \in \Omega$ such that $\sigma(a) = a + \varphi(\sigma)$, $\sigma(b) = b + \psi(\sigma)$ for every $\sigma \in \Pi$; whence we have

$$\varphi \longmapsto \Omega^{\mathrm{Ker}(\varphi)} = ([X, a^p - a])' \longmapsto ([X, a^p - a]) = h(\varphi)$$

and $([Y, b^p - b]) = h(\psi)$. Since $\sigma(a+b) = \sigma(a) + \varphi(\sigma) + \sigma(b) + \psi(\sigma) = \sigma(a+b) + (\varphi + \psi)(\sigma)$, it follows that $h(\varphi + \psi) = ([Z, (a+b)^p - (a+b)]) = ([X, a^p - a]) + ([Y, b^p - b]) = h(\varphi) + h(\psi)$. Hence h is a homomorphism, and so an isomorphism.

We conclude this section with a corollary, which is the well known description of normal separable extensions of degree p of a field of characteristic p.

Corollary 2.8. *Let R be a ring without proper idempotents. Then we have group isomorphisms*

$$\mathrm{Hom}_c(\Pi, G) \cong E(GR) \cong R^+ / \{r^p - r \,;\, r \text{ in } R\} \cong H^2(R, G)$$

where $H^2(R, G)$ is the second cohomology group in the sense of Harrison (cf. [1]).

Proof. Let $\bar{X} = X + (X^p - X - r)$ be in $[X, r]$. Considering the difference sequences of $\bar{X}^{p-1}, \sigma(\bar{X}^{p-1}), \cdots, \sigma^{p-1}(\bar{X}^{p-1})$, we can easily see that $\sigma(\bar{X}^{p-1}) - \bar{X}^{p-1}, \sigma(\sigma(\bar{X}^{p-1}) - \bar{X}^{p-1}) - (\sigma(\bar{X}^{p-1}) - \bar{X}^{p-1}), \cdots\cdots$ generate $[X, r]$, namely, $\bar{X}^{p-1}$ generates a G-normal basis (see also [10, Th. 4.1(b)]). Therefore the corollary is an immediate consequence of Th. 2.4 and Th. 2.7 and [1, Cor. 2.16(b)] (or [10, Th. 2.2]).

Remark 2.9. $E(GR)$ is isomorphic to the group $T(G, R)$ defined on [4, p. 3]. If $G = (\sigma_1) \times \cdots \times (\sigma_k)$ where (σ_i) is a cyclic group of order p, then by [3, Th. 3.11], we have

$$E(GR) \cong E((\sigma_1)R) \times \cdots \times E((\sigma_k)R).$$

Hence, the group $E(GR)$ of abelian $(p, \cdots, p)$-extensions of R in the sense of [8, p. 88] is completely determined by the group $E((\sigma)R)$ of cyclic p-extensions of R, where (σ) is a cyclic group of order p.

3. A group of strongly cyclic extensions. Throughout this section R will be a commutative ring which contains a primitive n-th root ζ of 1 such that n and $\{1-\zeta^i; i=1, 2, \cdots, n-1\}$ are in $U(R)$, the set of all inversible elements of R, and G will be a finite group of order n. In [6, Lemma 3.2], it is shown that for a cyclic extension A of R with Galois group G, if A has a G-normal basis then A is a strongly cyclic n-extension in the sense of [9, Def. 1.1].

The converse is also true.

Lemma 3.1. *Let A be a cyclic extension of R with a Galois group G. Then, A has a G-normal basis if and only if A is a strongly cyclic n-extension.*

Proof. Let A be a strongly cyclic n-extension of R. Then by [9, Th. 1.2], A is isomorphic to $R[X]/(X^n-u)$ for some $u\in U(R)$ (as Galois extension). Set $a=\sum_{i=0}^{p-1}\zeta^i\overline{X}^i$. Since $\{a, \sigma(a), \cdots, \sigma^{n-1}(a)\}$ generates $R[X]/(X^n-u)$ as R-module, A has a G-normal basis.

Now, a Galois GR-object A will be called a *strongly Galois GR-object* if A is strongly cyclic in the sense of [9, Def. 1.1]. Moreover, we denote $SE(GR)$ the set of GR-isomorphism classes of strongly Galois GR-objects, which is a subset of $E(GR)$. The following lemma is an immediate consequence of [9, Th. 1.1 and Th. 1.2] and the proof of Th. 1.2.

Lemma 3.2. *Let u be an arbitrary element in $U(R)$. Then $(X, u)=R[X]/(X^n-u)$ is a strongly Galois GR-object with the structure map $\alpha:(X, u)\longrightarrow(X, u)\otimes GR$ which is defined by $\alpha(\overline{X})=\sum_{i=0}^{n-1}\zeta^i\overline{X}\otimes v_i$, where $\overline{X}=X+(X^n-u)$. Moreover, if A is an arbitrary strongly Galois GR-object, then A is isomorphic to $R[X]/(X^n-u)$ as GR-object for some u in $U(R)$. Therefore we may write $A=(X, u)$.*

Replacing z in the proof of Th. 2.2 by $\sum_{i=0}^{n-1}\overline{X}\otimes\overline{Y}\otimes\zeta^i v_i$, we can prove the following

Theorem 3.3. *Let $A=(X, u)$ and $B=(Y, v)$ be strongly Galois GR-objects. Then $\widetilde{J}(A\otimes B)=(Z, uv)$.*

Let (X, u) be a strongly Galois GR-object. Then the map $\psi_r:(X, u)\longrightarrow GR$ is isomorphic as GR-object if and only if $\psi_r(\overline{X})=\sum_{i=0}^{n-1}r\zeta^i v_i$ and $u=r^n$, r in R. Then by Lemma 3.2 and Th. 3. 3, we have the following

Theorem 3.4. *$SE(GR)$ is a subgroup of $E(GR)$ and is isomorphic* *to* $U(R)/U(R)^n$ *as group.*

The following corollary is proved in a similar way as in the proof of Th. 2.7.

Corollary 3.5. *Let* R *be a ring without proper idempotents and* G *a group of prime order* q. *Then* $SE(GR)$ *is isomorphic to a subgroup of* $\mathrm{Hom}_c(\Pi, G)$, *where* Π *is the group of* R*-algebra automorphisms of* Ω, *a separable closure of* R.

Finally, we have

Theorem 3.6. *Let* R *be a ring without proper idempotents, and assume that every* G*-Galois extension of* R *is strongly cyclic* (*this will be the case if, for example,* R *is a semilocal ring, see* [2, Th. 4. 2]). *Then*

$$\mathrm{Hom}_c(\Pi, G) \cong SE(GR) = E(GR) \cong U(R)/U(R)^q \cong H^2(R, G).$$

Proof. This is an immediate consequence of Th. 3.4 and [10, Th. 2.2].

Remark 3.7. Let R be as in Th. 3.6. Then $E(GR)$ is isomorphic to the group $T(G, R)$ defined on [4, p. 3]. If $G = (\sigma_1) \times \cdots \times (\sigma_k)$ where (σ_i) is a cyclic group of order n, then by [3, Th. 3.11] we have

$$E(GR) \cong E((\sigma_1)R) \times \cdots \times E((\sigma_k)R).$$

Hence, the group $E(GR)$ of strongly abelian $(\sigma_1, \cdots, \sigma_k;\ n, \cdots, n)$-extensions of R in the sense of [9, p. 99] is completely determined by the group $E((\sigma)R)$ of strongly cyclic n-extensions of R.

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