A NOTE ON QUOTIENT RINGS OVER FROBENIUS EXTENSIONS

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In his paper [9]. Utumi introduced the concept of a maximal quotient ring which was related to Johnson [3]. As was shown there, a maximal quotient ring of a ring is determined up to isomorphism over the ring, and is called Utumi's ring of quotients of the ring. Indeed it is realized as the double centralizer of the injective hull of the ring viewed as a module over itself (see Lambek [5]).

While the notion of a Frobenius extension was introduced by Kasch [4]. Recently several results concerning Frobenius extensions, among others a commutor theory of Frobenius extensions, have been obtained in Miyashita [6].

The purpose of the present paper is to show the following theorem using the above results.

Theorem. *Let A/B be a Frobenius extension such that A is finitely generated as a B-module by elements which centralize B, and let $Q(A)$, $Q(B)$ be Utumi's rings of right quotients of A, B respectively. Then $Q(A)$ is a Frobenius extension of $Q(B)$ such that $Q(A)$ is finitely generated as a $Q(B)$-module by elements which centralize $Q(B)$, and that $Q(A)\cong Q(B)\otimes_B A\cong A\otimes_B Q(B)$ canonically. In particular $Q(B)=B$ implies $Q(A)=A$, and furthermore if A is generator as a left or a right B-module then the converse holds.*

It is interesting to observe that this theorem is applicable to Frobenius algebras, matrix rings and group rings over a finite group.

Throughout this note we assume that every ring has an identity, a subring of a ring has the common identity of the ring and that every module is unitary.

Let ${}_AM_{A'}$, ${}_AN_{A'}$ be A-A'-modules. We denote by M^n the direct sum of n-copies of M. If M is isomorphic to a direct summand of N^n for some positive integer n as A-A'-modules then we write ${}_AM_{A'}\,|\,{}_AN_{A'}$. Further it is written as ${}_AM_{A'}\sim{}_AN_{A'}$ if ${}_AM_{A'}\,|\,{}_AN_{A'}$ and ${}_AN_{A'}\,|\,{}_AM_{A'}$. Finally for a right R-module X, we denote its injective hull by $E(X_R)$.

1. Proof of Theorem. We begin with recalling the definition of a

Frobenius extension. Following Kasch [4], a ring extension A/B is called a *Frobenius extention* if A_B is finitely generated projective and ${}_BA_A \cong {}_B\mathrm{Hom}(A_B, B_B)_A$. As is known this is equivalent to that there exist a B-B-homomorphism h of A to B and $l_1, \cdots, l_n, r_1, \cdots, r_n$ in A such that $x = \sum r_i h(l_i x) = \sum h(xr_i) l_i$ for all x in A. When this is the case we shall call such h a *Frobenius homomorphism* and $(h : l_i, r_i)_i$ a *Frobenius system* for A/B. Let $(h : l_i, r_i)$ be a Frobenius system for A/B and X any right A-module. Then with respect to this system, there is the so-called trace homomorphism Tr : $\mathrm{End}(X_B) \to \mathrm{End}(X_A)$ defined by $(Tr(f))(x) = \sum (f(xr_i)) l_i$, x in X and f in $\mathrm{End}(X_B)$ ([7]).

Before going to prove the theorem, we prepare lemmas which are well known or easily seen.

Lemma 1. *Let A/B be a ring extension.*

(1) *If X_B is injective then* $\mathrm{Hom}(A_B, X_B)_A$ *is injective.*

(2) *If X_A is injective and ${}_BA$ is flat then X is injective as a B-module.*

Lemma 2. *Let A/B be a ring extension such that A is finitely generated as a B-module by elements which centralize B. If $f: X_B \to Y_B$ is an essential monomorphism (i. e. f is monic and Imf is essential in Y) then so is* $\mathrm{Hom}(A, f)$: $\mathrm{Hom}(A_B, X_B) \to \mathrm{Hom}(A_B, Y_B)$ *as right* (B- and *hence*) *A-modules (see*[1]).

Lemma 3. *Let X_R, Y_R be right R-modules such that $X_R \sim Y_R$, and let $u_1, \cdots, u_t \in$* $\mathrm{Hom}(Y_R, X_R)$, *$v_1, \cdots, v_t \in$* $\mathrm{Hom}(X_R, Y_R)$ *be a system with $\sum v_i u_i = Id_Y$. Then $\rho : D(X_R) \to D(Y_R)$ defined by $\rho(g) = \sum v_i . g . u_i$, g in $D(X_R)$, is a ring isomorphism over R, where $D(*)$ denotes the double centraliyer of $*$, and every homomorphism operates on the left side. Furthermore ρ is independent of the choice of (u_i, v_i) (see e. g.* [2]).

We are now ready to prove the theorem.

Proof of Theorem. *Let* $E(B)$ be the injective hull of B_B. By our assumptions, we have an exact sequence $B^n \to A \to 0$ as B-B-modules for some n, which induces an exact sequence $0 \to \mathrm{Hom}(A_B, E(B)_B) \to \mathrm{Hom}(B^n{}_B, E(B)_B) \cong E(B)^n$ as B-modules. Hence we have $\mathrm{Hom}(A_B, E(B)_B) \mid E(B)_B$. But obviously we have $E(B)_B \mid E(A_B)_B$. Furthermore by Lemmas 1 and 2, we have $E(A_B) \cong \mathrm{Hom}(A_B, E(B)_B)$ and $E(A_A) \cong \mathrm{Hom}(A_B, E(B)_B)$. It follows, recalling $E(B) \otimes_B A_A \cong \mathrm{Hom}(A_B, E(B)_B)_A$, that $E(B)_B \sim \mathrm{Hom}(A_B,$

$E(B)_B)_B$ and so $E(B)\underset{B}{\otimes}A_A \sim E(B)\underset{B}{\otimes}A\underset{B}{\otimes}A_A$. Therefore from [6, Th. 2. 10], $D(E(B)\underset{B}{\otimes}A_A)/D(E(B)\underset{B}{\otimes}A_B)$ is a Frobenius extension with Frobenius system $(h : l_i, r_i)_i$ such that $h(A)\subset B$, $\{l_i, r_i\}_i\subset A$ and $D(E(B)\underset{B}{\otimes}A_A)\cong D(E(B)\underset{B}{\otimes}A_B)\underset{B}{\otimes}A\cong A\underset{B}{\otimes}D(E(B)\underset{B}{\otimes}A_B)$ canonically, here it is to be noted that for every faithful B-module X_B, $\mathrm{Hom}(A_B, X_B)_A$ is faithful, since A/B being a Frobenius extension. Now we shall show $D(E(B)\underset{B}{\otimes}A_A)=Q(A)$ and $D(E(B)\underset{B}{\otimes}A_B)=Q(B)$. The former is evident because $E(B)\underset{B}{\otimes}A_A\cong E(A_A)$ (see [5]). The latter is also clear from Lemma 3, since $E(B)_B\sim E(B)\underset{B}{\otimes}A_B$. Finally we show that $V_A(B)$, the centralizer of B in A, is contained in $V_{Q(A)}(Q(B))$. Let x, y be arbitrary elements in $Q(B)$, $V_A(B)$ respectively. Then one can see $h(z(xy-yx))b=0$ for all z in $Q(A)$ and b in $x^{-1}B$, where $x^{-1}B=\{w\in B \mid xw\in B\}$. Since $x^{-1}B$ is a dense right ideal of B, $h(z(xy-yx))=O$, so that $xy-yx=O$ (see [5]). Thus recalling [6, Lemma 2.5], the proof is complete.

Remarks. 1) $Q(A)$ is right artinian (resp. right noetherian) if and only if so is $Q(B)$ (see [1]).

2) Suppose A to be generator as a left or a right B-module. If B has a right artinian right classical quotient ring $Q_{cl}(B)$ then A has a right artinian right clssical quotient ring $Q_{cl}(A)$ which is a Frobenius extension of $Q_{cl}(B)$ such that $Q_{cl}(A)$ is finitely generated as a $Q_{cl}(B)$-module by elements which centralize $Q_{cl}(B)$ and that $Q_{cl}(A)\cong Q_{cl}(B)\underset{B}{\otimes}A\cong A\underset{B}{\otimes}Q_{cl}(B)$ canonically. Indeed by Tachikawa [8, Th. 3.1], $Q_{cl}(B)\cong D(W_B)$ for a suitable faithful injective right B-module W. Since $W_B \ni w\mapsto(a\mapsto wh(a))\in\mathrm{Hom}(A_B, W_B)_B$ is monic from the assumption A being generator as a left or right B-module, the former is isomorphic to a direct summand of the latter that is isomorphic to $A\underset{B}{\otimes}W_B$. It follows by the same argument as above that $D(W\underset{B}{\otimes}A_A)/D(W\underset{B}{\otimes}A_B)$ is a Frobenius extension of the same type as in Theorem, and $D(W\underset{B}{\otimes}A_B)\cong D(W_B)$ over B. Furthermore it is easy to see that $D(W\underset{B}{\otimes}A_A)$ is a right artinian right classical quotient ring of A.

3) If A is a semi-prime right Goldie ring then so is B. As is known a group rong $R[G]$ over finite group G is a semi-prime right Goldie ring if and only if R is a semi-prime right Goldie ring and the order of G is non-zero-divisor in R.

Corollary 1. *Let A/B is a Frobenius extension with ${}_BA_B|{}_BB_B$. Then $Q(A)/Q(B)$ is a Frobenius extension with ${}_{Q(B)}Q(A)_{Q(B)}|{}_{Q(B)}Q(B)_{Q(B)}$ such that $Q(A)\cong Q(B)\otimes_B A\cong A\otimes_B Q(B)$ canonically. Moreover $Q(A)=A$ is equivalent to $Q(B)=B$.*

Proof. From [2, Prop. 5.6], ${}_BA_B|{}_BB_B$ implying ${}_BA_B\sim{}_BB_B$, this is a direct consequence of the above theorem.

Remark. This corollary is applicable to $(R)_n/R$, a matrix ring over R, $R[G]$, a group ring of finite group G over R, and a Frobenius algebra A/R over R. As a consequence of it, one obtains the well known facts $Q((R)_n)=(Q(R))_n$, $Q(R[G])=Q(R)[G]$ and $Q(A)=Q(R)\otimes_R A=A\otimes_R Q(R)$.

Corollary 2. *Let A/B be a Frobenius extension with ${}_AA\otimes_B A_A|{}_AA_A$ and A_B generator. Then $Q(A)/Q(B)$ is a Frobenius extension.*

Proof. First we note that B' is a Frobenius extension of A, where B' is the endomorphism ring of A_B. By the assumption, we have ${}_AB'_A\cong {}_AA\otimes_B\mathrm{Hom}(A_B, B_B)_A\cong{}_AA\otimes_B A_A|{}_AA_A$, so that from the above corollary, $Q(B')/Q(A)$ is a Frobenius extension and $Q(B')\cong Q(A)\otimes_A B'$ canonically. It follows that $Q(B')\otimes_{Q(A)}Q(B')\otimes_{B'}A\cong Q(B')\otimes_{B'}A\sim Q(B')$ as left $Q(B')$-modules since ${}_{B'}A\sim{}_{B'}B'$. Hence $\mathrm{End}({}_{Q(A)}Q(B')\otimes_{B'}A)$ is a Frobenius extension of $\mathrm{End}({}_{Q(B')}Q(B')\otimes_{B'}A)$ by [6, Th. 2.10]. Furthermore one obtains that $Q(B')/B'\sim\mathrm{End}({}_{Q(B')}Q(B')\otimes_{B'}A)/B$ relative to ${}_{B'}A_B$ and $Q(A)/A\sim\mathrm{End}({}_{Q(A)}Q(B')\otimes_{B'}A)/A$ relative to ${}_AA_A$, since $Q(B')\otimes_{B'}A\cong Q(A)\otimes_A B'\otimes_{B'}A\cong Q(A)\cong Q(A)\otimes_A A$ as left $Q(A)$-modules (for the notation "$*/*\sim*/*$" see § 2 below). Thus Prop. 2 in § 2 below makes the proof complete.

Corollary 3. *Let $A\supset T\supset B$ be rings. If A/B, T/B are Frobenius extensions of the same type as in Theorem, and if A/T is a Frobenius extension then so are $Q(A)\supset Q(T)\supset Q(B)$. Furthermore $Q(A)\cong Q(T)\otimes_T A\cong A\otimes_T Q(T)$ and $Q(T)\cong Q(B)\otimes_B T\cong T\otimes_B Q(B)$ canonically.*

Proof. Let $E(B)$ be the injective hull of B_B. Then by Lemma 2, we have $E(A_T)\cong\mathrm{Hom}(A_B, E(B)_B)_T(\cong E(B)\otimes_B A_T)$. But obviously we have $E(T_T)|E(A_T)$, and $E(A_T)|E(T_T)$, since $A_T|T_T$. It follows that $E(T_T)\sim E(A_T)\cong E(B)\otimes_B A_T$, and so $D(E(B)\otimes_B A_T)=Q(T)$. Further from the same argument in proving the theorem, we have $Q(A)=D(E(B)\otimes_B A_A)/Q(B)=D(E(B)\otimes_B A_B)$ a Frobenius extension of the same type as in Theorem such that $Q(A)\cong Q(B)\otimes_B A\cong A\otimes_B Q(B)$ canonically. Similarly we have that $Q(T)/Q(B)$ is a Frobenius extension of the same type as in Theorem such that $Q(T)\cong Q(B)\otimes_B T\cong T\otimes_B Q(B)$ canonically. Thus $Q(A)\cong Q(T)\otimes_T A\cong A\otimes_T Q(T)$ canonically. Now let $(h:l_i, r_i)_i$ and $(\alpha:x_j, y_j)_j$ be Frobenius systems for $Q(A)/Q(B)$ and A/T, respectively, such that $h(A)\subset B$ and $\{l_i, r_i\}_i\subset A$. Now we define a homomorphism α' of $Q(A)$ to $Q(T)$ by the composite homomorphism $Q(A)\cong Q(T)\otimes_T A\xrightarrow{1\otimes\alpha} Q(T)\otimes_T T\cong Q(T)$, that is, $\alpha'(x)=\sum h(xr_i)\alpha(l_i)$ $(x\in Q(A))$, where each isomorphism is canonical. Evidently α' is a left $Q(T)$-homomorphism. Further noting $Q(A)=A\cdot Q(B)=Q(B)\cdot A$ the quotientness $Q(T)$ shows α' a right $Q(T)$-homomorphism. Moreover $(\alpha:x_j, y_j)$ being a Frobenius system for A/T, it is easy to see that $(\alpha':x_j, y_j)$ is a Frobenius system for $Q(A)/Q(T)$. This completes the proof.

Remark. $R[G]\supset R[H]\supset R$ (H is a subgroup of a finite group G) satisfy the assumption in the above corollary. Indeed $Q(R)[G]\supset Q(R)[H]\supset Q(R)$.

2. Supplementary. The following proposition is a partial converse of [6, Th. 2.10].

Proposition 1. *Let A/B be a Frobenius extension having $(h:l_i, r_i)_i$ as its Frobenius system, M a right A-module which is generator, and $Tr: B'=\mathrm{End}(M_B)\longrightarrow A'=\mathrm{End}(M_A)$ the trace homomorphism of M with respect to this system. Then Tr is a Frobenius homomorphism if and only if $M\otimes_B A_A\,|\,M_A$.*

Proof. The if part is shown in [6, Th. 2.10] without the assumption M being generator. We shall show the only if part. Let $(Tr:f_j, g_j)_j$ be a Frobenius system for B'/A', so that $b'=\sum g_j Tr(f_j b')$ for all b' in B'. But for every u in $\mathrm{Hom}(M_A, A_A)$ and m in M, it is clear to see that $\lambda_m.\,h.\,u$ is an element in B', where $\lambda_m: B\ni b\rightarrow mb\in M$. It follows that $mh(u(m_1))=(\lambda_m.\,h.\,u)(m_1)=(\sum g_j Tr(f_j\lambda_m.\,h.\,u))(m_1)=\sum g_j(f_j(m)\,h(u(m_1)))$ $(m_1\in M)$. Thus M_A being generator we have $M\otimes_B A_A\,|\,M_A$ by [6, Prop. 2.3].

Following [6], a module ${}_AM_{A'}$ is called a *Morita module* if $M_{A'}$ is progenerator and $A=\mathrm{End}(M_{A'})$. Furthermore a ring extension A/B is said to be *Morita equivalent* to a ring extension A'/B' if there are Morita modules ${}_AM_{A'}$ and ${}_BN_{B'}$ such that $A\otimes_B N\cong M$ as left A-right B'-modules. In this case we write as $A/B\sim A'/B'$ relative to ${}_BN_{B'}$.

Now we shall show the following proposition which has been used to prove Cor. 2 in §1.

Proposition 2. *If* $Q(B)/B\sim A'/B'$ *then* $A'=Q(B')$.

Before proving this, we need a lemma.

Lemma. *Let* ${}_AM_{A'}$ *be a Morita module and* ${}_{A^*}X_A$ *an* A^*-A-*module with* $A^*=\mathrm{End}(X_A)$, $A=\mathrm{End}({}_{A^*}X)$. *Then there is a ring isomorphism* $\rho: A'\to \mathrm{End}({}_{A^*}X\otimes_A M)$ *defined by* $\rho(a')(x\otimes m)=x\otimes m^{a'}$, *where* $m^{a'}$ *denotes the image of* m *by* a' $(a'\in A', m\in M, x\in X)$.

Proof. Since ${}_AM$ is finitely generated projective we have the canonical isomorphism $M\ni m\longmapsto (x\longmapsto x\otimes m)\in \mathrm{Hom}({}_{A^*}X, {}_{A^*}X\otimes_A M)$ as A-A'-modules. Then this induces the isomorphism $\mathrm{Hom}({}_AM, {}_AM)\longrightarrow \mathrm{Hom}({}_AM, {}_A\mathrm{Hom}({}_{A^*}X, {}_{A^*}X\otimes_A M))$. Further the latter is canonically isomorphic to $\mathrm{Hom}({}_{A^*}X\otimes_A M, {}_{A^*}X\otimes_A M)$. Thus this completes the proof.

Proof of Proposition 2. Let $E(B)$ be the injective hull of B_B, and let ${}_BN_{B'}$, ${}_{Q(B)}M_{A'}$ be Morita modules as above. Then we may identify $Q(B)\otimes_B N$ with M. Since ${}_BN_{B'}$ is a Morita module, we have $E(B)\otimes_B N\cong E(B\otimes_B N_{B'})\cong E(N_{B'})\sim E(B'_{B'})$ as right B'-modules, and so $D(E(B)\otimes_B N_{B'})\cong D(E(B'_{B'}))$ over B', that is $D(E(B)\otimes_B N_{B'})=Q(B')$. On the other hand, recalling ${}_{B^*}E(B)\otimes_B N_{B'}\cong {}_{B^*}E(B)\otimes_{Q(B)} Q(B)\otimes_B N_{B'}$ (canonical), the above lemma yields that $\mathrm{End}({}_{B^*}E(B)\otimes_B N)\cong \mathrm{End}({}_{B^*}E(B)\otimes_{Q(B)} Q(B)\otimes_B N)\cong \mathrm{End}({}_{Q(B)}Q(B)\otimes_B N)$ over B', where $B^*=\mathrm{End}(E(B)_B)$. Furthermore once again using the above lemma, we have $D(E(B)\otimes_B N_{B'})\cong \mathrm{End}({}_{B^*}E(B)\otimes_B N)$ over B'. It follows that $A'=\mathrm{End}({}_{Q(B)}Q(B)\otimes_B N)=Q(B')$. Thus the proof is complete.

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(*Received March 10, 1972*)

Added in proof. After submitting this paper, I have found that Proposition 2 was already obtained by Darrell R. Turnidge in a different point of view in Theorem 3.1 of his paper: Torsion theories and rings of quotients of Morita equivalent rings, Pacific J. Math. 37 (1971), 225-234.