ANOTHER PROOF OF THE INVARIANCE OF ULM'S FUNCTIONS IN COMMUTATIVE MODULAR GROUP RINGS

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In this note we give a short and natural proof of the following theorem due to Berman and Mollov [1] and May [2].

Theorem. *Let Z_pG be the group ring of a p-primary group G over Z_p, the field of p elements. Suppose $\theta : Z_pG \simeq Z_pH$. Then G and H have the same Ulm's functions.*

The proof is a direct consequence of a lemma of Jennings [3] which we give below. First, we need some notation. We write all groups multiplicatively and define $G^p = \{g \in G \mid g = x^p \text{ for some } x \in G\}$. Inductively, for ordinals β we have

$$G^{p^{\beta+1}} = \left(G^{p^\beta}\right)^p \quad \text{and} \quad G^{p^\beta} = \bigcap_{\alpha<\beta} G^{p^\alpha}$$

for β a limit ordinal.

If K is a subgroup of G, by $\triangle(G;K)$ we mean the ideal of Z_pG generated by elements of the form $1-k$, $k \in K$. Sometimes we write $\triangle(K)$ if the context is clear. We denote $\{x \in K \mid x^p = 1\}$ by $K[p]$.

Lemma. *Let G be a p-primary abelian group and N a subgroup. Then*

(1) $G/G^p \simeq \triangle(G)/\triangle^2(G)$, *and*

(2) $N/N^p \simeq \triangle(G;N)/\triangle(G)\cdot\triangle(G;N)$.

Proof. Define $\lambda : G \to \triangle(G)/\triangle^2(G)$ by $\lambda(g) = g-1+\triangle^2(G)$. Since

(*) $g_1g_2-1 = (g_1-1)+(g_2-1)+(g_1-1)(g_2-1)$,

λ is an epimorphism with kernel $= \{g \in G \mid g-1 \in \triangle^2(G)\} = G^p$ by Jennings [3], proving (1). Actually, Jennings proved this equality for finite groups but since in an equation $g-1=\delta \in \triangle^2(G)$, only a finite number of elements of G occur, his result is applicable to our case.

For the second part, define

$$\mu : N \to \triangle(G;N)/\triangle(G)\cdot\triangle(G;N) \text{ by } \mu(n) = \overline{n-1}.$$

It follows from (*) and

$$g(n-1)=n-1+(g-1)(n-1)$$

that μ is an epimorphism with kernel $=\{n \mid n-1\in\triangle(G)\cdot\triangle(G;N)\}$. It remains to prove:

$$(**)\qquad n-1\in\triangle(G)\cdot\triangle(G;N)\Rightarrow n\in N^{p}.$$

Choose a transversal $\{g_i\}$ of N in G with $g_1=1$. Define for $g_i n\in G$, $\sigma(g_i n)=n$ and extend this linearly to $\sigma: Z_pG\to Z_pN$. Now,

$$n-1=\sum_i \gamma_i(n_i-1),\ \gamma_i\in\triangle(G),\ n_i\in N.$$

Therefore

$$n-1=(n-1)^{\sigma}=\sum_i \gamma_i^{\sigma}(n_i-1)\ \text{ and }\ n-1\in\triangle^2(N;N).$$

Hence, $n\in N^p$. This proves (**) and therefore (2).

Remark. The above lemma is a special case of a similar result that holds for arbitrary (not necessarily abelian or finite) groups. Also, there is a corresponding result for integral group rings (see, Sehgal [4]). For the purpose of this paper the above will suffice.

Proof of Theorem. Now, suppose $\theta: Z_pG\simeq Z_pH$. We may assume here that θ is normalized; if $\theta(g)=\sum_{h\in H}\alpha_h h$ then $\sum_{h\in H}\alpha_h=1$. By noting that $\theta(g^p)=(\sum_{h\in H}\alpha_h h)^p=\sum_{h\in H}\alpha_h^p h^p$, we have that θ maps Z_pG^p isomorphically onto Z_pH^p. By a simple induction

$$\theta: Z_pG^{p^\beta}\simeq Z_pH^{p^\beta}\qquad \text{for all ordinals } \beta.$$

We show first that the finite Ulm invariants are equal. The ith Ulm invariant, $i<\omega$ (the first limit ordinal), is the dimension of $(G^{p^i})[p]/(G^{p^{i+1}})[p]$. For convenience let us denote $(G^{p^i})[p]$ by L_i.

By the lemma we have an isomorphism

$$L_i\simeq\triangle(G;L_i)/\triangle(G)\cdot\triangle(G;L_i).$$

Under θ, $\triangle(L_i)$ is isomorphic to $\triangle(M_i)$ where $M_i=(H^{p^i})[p]$. Thus we obtain for each i the commutative diagram below:

$$\begin{array}{ccccccc}
L_i & \simeq & \triangle(L_i)/\triangle(G)\triangle(L_i) & \simeq & \triangle(M_i)/\triangle(H)\triangle(M_i) & \simeq & M_i\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
L_{i+1} & \simeq & \triangle(L_{i+1})/\triangle(G)\triangle(L_{i+1}) & \simeq & \triangle(M_{i+1})/\triangle(H)\triangle(M_{i+1}) & \simeq & M_{i+1}
\end{array}$$

and thus $L_i/L_{i+1}\simeq M_i/M_{i+1}$.

The observation that $Z_pG^{p^\beta}\simeq Z_pH^{p^\beta}$ allows us to conclude that even the transfinite Ulm invariants are equal.

REFERENCES

[1] S. D. BERMAN and T. ZH. MOLLOV: On group rings of abelian p-groups of any cardinality, Mat. Zam **6** (1969), 381—392 (Translation Math. Notes **6** (1969), 686—692).

[2] W. MAY: Commutative group algebras, Trans. Amer. Math. Soc. **136** (1969), 139—149.

[3] S. A. JENNINGS: The structure of the group rings of a p-group over a modular field, Trans. Amer. Math. Soc. **50** (1941), 175—185.

[4] S. K. SEHGAL: On the isomorphism of integral group rings II, Can. J. Math. **21** (1969), 1182—1188.

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