ON THE BIMODULE STRUCTURE OF GALOIS EXTENSIONS

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Throughout the present note, A will represent *an (Artinian) simple ring which is finite Galois over a simple subring B.* Then, it is known that A is $B\cdot V_A(B)$-A-irreducible. (See [4]. As to terminologies used without mention, we follow [4].) We use the following notations: $C=V_A(A)$, $Z=V_B(B)$, $V=V_A(B)$, $H=V_A(V)$, $C_0=V_V(V)=V\cap H$, $G=$ the Galois group of A/B, $\{\sigma_1=1, \sigma_2, \cdots, \sigma_n\}$ is a (fixed) representative system of G modulo the normal subgroup I consisting of all inner automorphisms, and $\bar{G}=G/I$, which may and will be regarded as the Galois group of H/B. We set further $S=V_l\cdot V_r$ and $R=\mathrm{Hom}\,({}_BA_B, {}_BA_B)$, where V_l (resp. V_r) the left (resp. right) multiplication of V. To be easily seen, $R=GV_r=\bigoplus_{i=1}^n \sigma_i S$ and $S\cong V^\circ\otimes_C V$ is a Frobenius ring, where V° is the opposite of V. (See, for instance [3; Lemma 3].)

In this note, the main theme of our discussion will concern the bimodule structure of ${}_BA_B$. We shall prove first that ${}_BA_B$ is a direct sum of local submodules, where a module ${}_BM_B$ is said to be *local* if it contains one and only one maximal submodule (Theorem 1). Next, we shall explain when ${}_BA_B$ is completely reducible (Theorem 2) or local (Theorem 3).

The next lemma will play an essential role in our subsequent study.

Lemma 1. *If T is a B-B-submodule of A then the restriction $T|R$ of R to T contains a free V_r-basis that forms at the same time a free A_r-basis of $T|GA_r$, $[T|R:V_r]=[T|GA_r:A_r]=[T:B]_l$ and $T|R=$* $\mathrm{Hom}\,({}_BT_B, {}_BA_B)$.

Proof. As is well-known, T is left (resp. right) B-free. Hence, the lemma is contained in [4; Lemma 5.8].

Theorem 1. *R is a Frobenius ring and ${}_BA_B$ is a direct sum of local modules.*

Proof. To be easily seen, the map h defined by $\sum_{i=1}^n \sigma_i s_i \mapsto s_1$ is a Frobenius homomorphism of the ring extension R/S, namely, R/S is a free Frobenius extension. Since S is a Frobenius ring, so is R by [1;

Satz 10]. Next, we shall prove the latter part. To our end, it is enough to show that if e is a primitive idempotent of R then ${}_BAe_B$ is local. Let T_1 and T_2 be arbitray proper B-B-submodules of Ae. Noting that $[T_i|R:V_r]=[T_i:B]_l<[Ae:B]_l=[Ae|R:V_r]$ (Lemma 1), the kernel of the restriction map $h_i: Ae|R\to T_i|R$ is non-zero. Since R is a Frobenius ring, $\ker h_1 \cap \ker h_2$ contains a submodule isomorphic to the unique minimal right subideal of eR. Hence, the kernel of the restriction map $Ae|R\to T_1+T_2|R$ is non-zero, which implies $T_1+T_2\neq Ae$ and that ${}_BAe_B$ is local.

Theorem 2. *The following conditions are equivalent*:

(1) ${}_BA_B$ *is completely reducible.*

(2) R *is semisimple.*

(3) a) C_0/C *is separable*;

b) $\sum_{i=1}^n c\sigma_i=1$ *for some* $c\in C_0$.

Proof. (1)$\Rightarrow$(2): This is evident by Lemma 1.

(2)$\Rightarrow$(1): It suffices to prove that if $eR\,(e^2=e)$ is a minimal right ideal then ${}_BAe_B$ is irreducible. Let T' be an arbitrary non-zero B-B-submodule of Ae. Noting that $Ae|R$ is R-irreducible, we obtain $[T':B]_l=[T'|R:V_r]=[Ae|R:V_r]=[Ae:B]_l$ (Lemma 1), namely, $T'=Ae$.

(2)$\Rightarrow$(3): Evidently, the semisimplicity of R implies the semisimplicity of S, equivalently, the separability of C_0/C. Since ${}_BA_B$ is then completely reducible, so is ${}_BH_B$. Accordingly, $\mathrm{Hom}({}_BH_B, {}_BH_B)=\bar{G}C_{0r}$ is semisimple. Hence, C_0 is a direct summand of H as $\bar{G}C_{0r}$-module. Now, noting that H contains an element a with $1=\sum_{\bar\sigma\in\bar G} a\bar\sigma=\sum_{i=1}^n a\sigma_i$, we readily obtain (iii) b).

(3)$\Rightarrow$(2): Let $\sum_{i=1}^n c\sigma_i=1$ for some $c\in C_0$. To be easily verified, $\sum_{i=1}^n \sigma_i^{-1}c_r\sigma_i=1$ and $\sum_{i=1}^n x\sigma_i^{-1}c_r\otimes\sigma_i=\sum_{i=1}^n \sigma_i^{-1}c_r\otimes\sigma_i x$ (in $R\otimes_S R$) for every $x\in R$. This means that R is a separable extension of the semisimple ring S. Then, R is semisimple by [2; Lemma 2.10 (1)].

Corollary 1. *If* C_0 *is separable over* C *and* n *is not divisible by* char A, *then* ${}_BA_B$ *is completely reducible.*

Theorem 3. *The following conditions are equivalent*:

(1) ${}_BA_B$ *is local.*

(2) R *is a local ring.*

(3) a) $V=Z$ *and is purely inseparable over* C;

b) *either A/B is inner Galois or $\bar{G}$ is a p-group and* char $A=p$.

Proof. (1)$\Rightarrow$(2): This is evident by Th. 1.

(2)$\Rightarrow$(3): Since R is local, so is the subring S which is isomorphic to $V^\circ\otimes_C V$. Accordingly, $C_0\otimes_C C_0$ is a local ring, namely, C_0/C is purely inseparable. If $[V:C_0]=m$ then $(C_0)_m(\cong V^\circ\otimes_{C_0}V)$ is a homomorphic image of the local ring S. Hence, it follows $V=C_0$. Further, C_0/Z is Galois with $C_0|\bar{G}$ as Galois group, and so if $[C_0:Z]=t$ then $(Z)_t$ ($\cong C_0|\bar{G}C_{0r}$) is homomorphic to R. It follows therefore $Z=C_0=V$ and $H|R$ is isomorphic to the group ring of $\bar{G}$ over Z. Hence, if $n>1$ then $\bar{G}$ is a p-group and $p=$char Z by [4; Lemma 13.4].

(3)$\Rightarrow$(2): Let ϕ be the ring homomorphism of the local ring $S=Z_l\cdot Z_r$ onto Z given by $\sum z_{il}\cdot z'_{ir}\mapsto\sum z_i z'_i$. Then, the kernel of ϕ is the radical J of S. Now, one will easily see that the kernel of the restriction map $R\rightarrow H|R$ is $\bigoplus_{i=1}^{n}\sigma_i J$ and nilpotent. Since $H|R=\bar{G}Z_r$ is a local ring again by [4; Lemma 13.4], we can easily see that R is a local ring.

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REFERENCES

[1] F. KASCH: Grundlagen einer Theorie der Frobeniuserweiterungen, Math. Ann. **127** (1954), 453—474.

[2] Y. MIYASHITA: Locally finite outer Galois theory, J. Fac. Sci. Hokkaido Univ., Ser. I, **20** (1967), 1—26.

[3] T. NAKAYAMA: On Frobeniusean algebras II, Ann. Math. **42** (1941), 1—21.

[4] H. TOMINAGA and T. NAGAHARA: Galois Theory of Simple Rings, Okayama Math. Lectures, Okayama Univ., 1970.

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