ON STRONGLY CYCLIC EXTENSIONS OF COMMUTATIVE RINGS

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In [6], we have studied cyclic extensions of commutative rings which contain the prime field $GF(p)$ $(p \neq 0)$. In this paper, we shall study strongly cyclic extensions of commutative rings and sharpen some results of Kishimoto for strongly abelian extensions [4]. In §1, we shall give some properties of strongly cyclic extensions. We consider first strongly cyclic extensions of an arbitrary commutative ring, and later place some restrictions on the base ring, e. g., without proper idempotents. Applying the results of §1, we shall study in §2 strongly abelian extensions.

In all that follows, B will be assumed to be a commutative ring which contains a primitive n-th root ζ of 1 such that n and $\{1-\zeta^i;\ i=1, 2, \cdots, n-1\}$ are in $U(B)$, the set of all unit elements of B. We set $\Gamma=\{1, \zeta, \cdots, \zeta^{n-1}\}$. Further, all ring extensions of B will be assumed to be commutative and have the identity 1. Moreover, we shall understand by a divisor of n a positive integer which divides n. As to other terminologies used in this paper, we follow [1], [3] and [6].

2. Strongly cyclic extensions.

We now begin with lemmas for separable polynomials.

Lemma 1.1. *Let m be a divisor of n, and γ_i a primitive m-th root of 1 in Γ. Let $f(X)=X^m-b \in B[X]$. Then $f(X)$ is separable if and only if b is a unit in B.*

Proof. We set $B[x]=B[X]/(X^m-b)$ where $x=X+(X^m-b)$. If $f(X)$ is separable then mx^{m-1} is a unit in $B[x]$ by [2, Prop. 1.8]. Therefore x is a unit in $B[x]$. Since $x^m=b$, b is a unit in $B[x]$. Hence noting that $1, x, \cdots, x^{m-1}$ is a free basis of $B[x]$ over B, it is easily seen that b is a unit in B. Conversely, if b is a unit in B then x is a unit in $B[x]$. Since $f(\gamma_i^i x)=0$ $(0 \leqq i \leqq m-1)$ and $\gamma_i^i x-\gamma_i^j x \in U(B[x])$ $(i \neq j)$, we have $f(X)=(X-x)(X-\gamma_i x)\cdots(X-\gamma_i^{m-1}x)$. Then it follows from [5, Th.] that $f(X)$ is separable.

Lemma 1.2. *Let m be a divisior of n, and η a primitive m-th root of 1 in Γ. Let $f(X)=X^m-b \in B[X]$ be separable. If there is a ring extension A of B such that $f(a)=0$ for some a in $U(A)$ then $f(X)=(X-a)(X-\eta a)\cdots(X-\eta^{m-1}a)$; and if, in addition, there exists a B-algebra automorphism σ of A with $\sigma(a)=\eta a$ then there exists an isomorphism $B[X]/(f(X)) \to B[a]$ such that*

$$(\mathrm{i}) \qquad g(X)+(f(X)) \to g(a).$$

Proof. The first assertion is clear by the proof of Lemma 1.1. The latter can be proved by making use of the same method as in the proof of [6, Lemma 1.1].

Definition 1.1. Let $A \supset T \supset B$ be ring extensions, and m a divisior of n. If there exists a unit a in A and a ring automorphism σ of A such that

(1) σ is of order m,

(2) T is the fixring of σ in A,

(3) $\sigma(a)=\eta a$ for some primitive m-th root η of 1 in Γ,

then A/T is a (σ)-Galois extension (see, the proof of Th. 1.2) and is called *a strongly cyclic $(\sigma; m; a)$-extension.* Occasionally, a strongly cyclic $(\sigma; m; a)$-extension will be called simply *a strongly cyclic $(\sigma; m)$-extension or a strongly cyclic m-extension.*

Lemma 1.3. *Let $A \supset T \supset B$ be ring extensions, and σ an automorphism of A of finite order m. If m is a divisor of n and T is the fixring of σ in A then the following conditions are equivalent.*

(1) *A/T is a strongly cyclic $(\sigma; m)$-extension.*

(2) *There exists a unit a in A such that $a+\eta\sigma(a)+\cdots+\eta^{m-1}\sigma^{m-1}(a) \in U(A)$ for some primitive m-th root η of 1 in Γ.*

Proof. Set $g(a)=a+\eta\sigma(a)+\cdots+\eta^{m-1}\sigma^{m-1}(a)$ where $\eta^m=1$. Then $\sigma(g(a))=\eta^{-1}g(a)$. If $\sigma(a)=\eta a$ then $g(a^{m-1})=ma^{m-1}$. This proves (1)$\Leftrightarrow$(2).

Theorem 1.1. *Let m be a divisor of n, and η a primitive m-th root of 1 in Γ. Assume $f(X)=X^m-b \in B[X]$ is separable. Then $B[X]/(f(X))$ is a strongly cyclic $(\sigma; m)$-extension of B where σ is given by*

$$(\mathrm{ii}) \qquad X+(f(X)) \to \eta X+(f(X)).$$

Proof. The map $\sum_i b_i X^i \to \sum_i b_i (\eta_i X)^i$ defines an automorphism of $B[X]$ sending $f(X)$ into $f(\eta_i X) = f(X)$, which induces therefore an automorphism σ of $B[X]/(f(X))$ of order m. Set $x = X + (f(X))$. Since x and $1-\eta^i$ $(1 \leqq i \leqq m-1)$ are in $U(B[x])$, the same argument as in the proof of [6, Th. 1.1] enables us to see that B is the fixring of σ in $B[x]$. Therefore $B[x]$ is a strongly cyclic $(\sigma; m)$-extension of B.

Combining Lemma 1.2 and Th. 1.1, we have the following

Corollary 1.1. *Let A be a ring extension of B, and σ a B-algebra automorphism of A. If there exists an element a of $U(A)$ such that $a^m \in B$ and $\sigma(a) = \eta_i a$ for some primitive m-th root η_i of 1 in Γ then the diagram*

$$\begin{array}{ccc} B[X]/(f(X)) & \xrightarrow{\text{(i)}} & B[a] \\ \text{(ii)}\downarrow & & \downarrow \sigma|B[a] \\ B[X]/(f(X)) & \xrightarrow{\text{(i)}} & B[a] \end{array} \quad ; \quad f(X) = X^m - a^m$$

is commutative, and $B[a]$ is a strongly cyclic $(\sigma|B[a]; m)$-extension of B.

Theorem 1.2. *If A is a strongly cyclic $(\sigma; m; a)$-extension of B then $f(X) = X^m - a^m$ is a separable polynomial in $B[X]$, $A = B[a]$ and is isomorphic to $B[X]/(f(X))$ by* (i).

Proof. Since A is a strongly cyclic $(\sigma; m; a)$-extension of B, $\sigma(a) = \eta_i a$ for some primitive m-th root η_i of 1 in Γ. Then $a^m = \sigma(a^m)$ is in $U(B)$ and $f(X)$ is separable by Lemma 1.1. By Lemma 1.2, $B[a]$ is isomorphic to $B[X]/(f(X))$ by (i). Now, it is easily seen that for $0 \leqq i, j, k \leqq m-1$, $\delta_{1,\sigma^k} = (\Pi_{i \neq j} (\eta^i a - \eta^j a))^{-1} \Pi_{i \neq j} (\eta^i a - \sigma^k(\eta^j a))$ which is written as $\sum_{t=1}^{s} x_t \sigma^k(y_t)$, $x_t, y_t \in B[a]$. Then for every $u \in A$, we have $u = \sum_{t=1}^{s} x_t T_{(\sigma)}(uy_t) \in B[a]$, where $T_{(\sigma)}(uy_t) = uy_t + \sigma(uy_t) + \cdots + \sigma^{n-1}(uy_t)$. Hence it follows that $B[a] = A$.

As a direct consequence of Th. 1.2, we have the following

Corollary 1.2. *Let A and A' be strongly cyclic $(\sigma; m; a)$ and $(\sigma'; m; a')$-extension of B, respectively. If $a^m = a'^m$ then A and A' are B-algebra isomorphic.*

Theorem 1.3. *Let T/B be a strongly cyclic $(\tau; m; t)$-extension, where $n = ms$. Then there is a strongly cyclic $(\sigma; n; a)$-extension A of B such that*

(1) *A is a strongly cyclic $(\sigma^m; s; a)$-extension of T.*
(2) *$\sigma|T=\tau^u$ for some positive integer u with $(u, m)=1$.*
(3) *$a^s=t$.*

Proof. By assumption, $\tau(t)=\eta t$, where $\eta=(\zeta^s)^i$ and $(i, m)=1$. There exists then a positive integer u with $(u, m)=1$ such that $\tau^u(t)=\zeta^s t$. To be easily seen, $T[X] \ni \sum_i t_i X^i \to \sum_i \tau^u(t_i)(\zeta X)^i \in T[X]$ is an automorphism of order n and it induces an automorphism σ in $A=T[X]/(X^s-t)$:

(iii) $$\sum_i t_i X^i+(X^s-t) \to \sum_i \tau^u(t_i)(\zeta X)^i+(X^s-t).$$

Then σ^m is a T-algebra automorphism of A:

(ii′) $$X+(X^s-t) \to \zeta^m X+(X^s-t).$$

Hence by Th. 1.1, A is a strongly cyclic $(\sigma^m; s)$-extension of T. We set $a=X+(X^s-t)$. Then, a is a unit of A with $a^s=t$ and $\sigma(a)=\zeta a$. Combining this with $\sigma|T=\tau^u$, it follows that A is a strongly cyclic $(\sigma; n; a)$-extension of B.

Theorem 1.4. *Let A be a strongly cyclic $(\sigma; n; a)$-extension of B, $n=ms$ and $t=a^s$. Assume T is the fixring of (σ^m) in A. Then A is a strongly cyclic $(\sigma^m; s; a)$-extension of T and T is a strongly cyclic $(\sigma|T; m; t)$-extension of B, and the following diagram*

$$\begin{array}{ccc} T[X]/(X^s-t) & \xrightarrow{\text{(i′)}} & A \\ \text{(iii)}\downarrow & & \downarrow \sigma \\ T[X]/(X^s-t) & \xrightarrow{\text{(i′)}} & A \end{array}$$

is commutative, where (i′) *is defined as for $B[X]/(f(X))$ in Lemma* 1.2.

Proof. It is clear that $\sigma|T$ is of order m and B is the fixring of $\sigma|T$ in T. Since $\sigma^m(t)=t$, t is in $U(T)$. We set $\sigma(a)=\zeta^i a$, where $(i, n)=1$. Then $(\sigma|T)(t)=\zeta^{is}t$, and T is a strongly cyclic $(\sigma|T; m; t)$-extension of B. Now the commutativity of the diagram will be easily seen.

Theorem 1.5. *Let A be a strongly cyclic $(\sigma; n)$-extension of B.*

(1) *If C is a B-algebra with an identity element then $C \otimes_B A$ is a strongly cyclic $(1 \otimes \sigma; n)$-extension of $C \otimes 1$ $(\cong C)$.*

(2) *If N is a proper ideal of B then $AN\cap B=N$ and A/AN ($\cong B/N\otimes_B A$) is a strongly cyclic $(1\otimes\sigma;\ n)$-extension of B/N.*

(3) *If S is a multiplicatively closed subset of B not containing 0 then the quotient ring $A[S^{-1}]$ ($\cong B[S^{-1}]\otimes_B A$) is a strongly cyclic $(1\otimes\sigma;\ n)$-extension of $B[S^{-1}]$.*

Proof. Since B is a direct summand of A as B-module, $C\cong C\otimes 1\subset C\otimes_B A$, and (1) is almost evident. (2) and (3) are direct consequences of (1).

The next will be easily verified.

Theorem 1.6. *Let m_1, m_2 be divisors of n with $(m_1, m_2)=1$. If A_i are strongly cyclic $(\sigma_i;\ m_i;\ a_i)$-extensions of B $(i=1, 2)$ then $A_1\otimes_B A_2$ is a strongly cyclic $(\sigma_1\otimes\sigma_2;\ m_1m_2;\ a_1\otimes a_2)$-extension of B.*

Theorem 1.7. *Let A be a strongly cyclic $(\sigma;\ m;\ a)$-extension of B, and set $m=q_1^{f_1}q_2^{f_2}\cdots q_t^{f_t}$, where $q_1, q_2, \cdots q_t$ are the prime divisors of m with $(q_i, q_j)=1$ $(i\neq j)$. Then A is isomorphic to $A_1\otimes_B A_2\otimes\cdots\otimes_B A_t$, where A_i is a strongly cyclic $q_i^{f_i}$-extension of B.*

Proof. We have $(\sigma)=(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_t)$, where (σ_i) is a cyclic group of order $q_i^{f_i}$. Let $\mathfrak{G}_i=(\sigma_1)\times\cdots\times(\sigma_{i-1})\times(\sigma_{i+1})\times\cdots\times(\sigma_t)$, A_i the fixring of $\mathfrak{G}_i$ in A, and $a_i=N_{\mathfrak{G}_i}(a)$ $(=\Pi_{\tau\in\mathfrak{G}_i}\tau(a))$. Then $a_i\in U(A_i)$ and $\sigma_i(a_i)=\sigma_iN_{\mathfrak{G}_i}(a)=N_{\mathfrak{G}_i}(\sigma_i(a))=\eta N_{\mathfrak{G}_i}(a)=\eta a_i$ for some primitive $q_i^{f_i}$-th root η of 1 in Γ. Therefore A_i is a strongly cyclic $q_i^{f_i}$-extension of B and we can prove easily $A\cong A_1\otimes_B A_2\otimes\cdots\otimes_B A_t$.

Lemma 1.4. *Let B be a ring without proper idempotents, and q a prime divisor of n. In order that a separable polynomial $f(X)=X^q-c$ is irreducible in $B[X]$ if and only if $f(b)\neq 0$ for every b in B.*

Proof. Since q is a prime divisor of n, the proof proceeds in the same way as in the proof of [6, Lemma 1.2].

Lemma 1.5. *Let B be a ring without proper idempotents, and $f(X)$ a separable polynomial in $B[X]$. Let $B[X]/(f(X))=A_1\oplus\cdots\oplus A_k$, where the A_i are ideals with no proper idempotents, and $f(X)=f_1(X)\cdots f_l(X)$, where the $f_j(X)$ are irreducible polynomials in $B[X]$. Then $k=l$.*

Proof. C. f., the proof of [3, Th. 2.9].

Theorem 1.8. *Let B be a ring without proper idempotents, and m a divisor of n. Then, there is a strongly cyclic $(\tau; m)$-extension of B without proper idempotents if and only if there exists an element b_0 in $U(B)$ such that X^m-b_0 is irreducible in $B[X]$.*

Proof. Let A be a strongly cyclic $(\tau; m; a)$-extension of B without proper idempotents. Then, by Th. 1.2, we have $a^m \in U(B)$ and a B-algebra isomorphism $B[X]/(X^m-a^m) \cong B[a] = A$. Therefore X^m-a^m is irreducible by Lemma 1.5. Conversely, if we set $A=B[X]/(X^m-b_0)$ then A is a strongly cyclic $(\tau; m)$-extension of A by Th. 1.1. Since X^m-b_0 is irreducible, A has no proper idempotents again by Lemma 1.5.

Now, we shall introduce here the following definition.

Definition 1.2. Let $f(X)$ be a monic polynomial in $B[X]$. A ring extension A of B is called *a splitting ring of $f(X)$* if $f(X)=(X-a_1)\cdots(X-a_k)$ and $A=B[a_1, \cdots, a_k]$ with some $a_i \in A$.

Theorem 1.9. *Let m be a divisor of n, and A a splitting ring of a separable polynomial $f(X)=X^m-c$ in $B[X]$. Assume that A has no proper idempotents and A is projective as B-module. Then A is a strongly cyclic extension of B. If, in particular, m is prime and $A \supsetneq B$ then A is a strongly cyclic m-extension of B.*

Proof. Since A is a splitting ring of X^m-c, we have $f(X)=(X-a_1)\cdots(X-a_m)$ and $A=B[a_1, \cdots, a_m]$ with some $a_i \in A$. Then by [3, Lemma 2.1], we obtain $\{a_1, \cdots, a_m\}=\{a=a_1, \eta a, \cdots, \eta^{m-1}a\}$ for a primitive m-th root η of 1 in Γ, and hence $A=B[a]$. By [5, Cor. 6], A is a Galois extension of B. Let $\mathfrak{G}$ be the Galois group of A/B, and $\sigma, \tau \in \mathfrak{G}$. If $\sigma(a)=\eta^i a$ and $\tau(a)=\eta^j a$ then $\sigma\tau(a)=\eta^i\eta^j a$. Hence $\mathfrak{G}$ is isomorphic to a subgroup of the cyclic group generated by η. This enables us to see that A is a strongly cyclic extension of B.

Lemma 1.6. *Let T be a ring extension of B, and τ a B-algebra automorphism of T whose order is a prime divisor q of n. Suppose there exist elements t, t_0 in $U(T)$ such that $\tau(t_0)=t_0t^q$ and $N_{(\tau)}(t)=\gamma$ for some primitive q-th root γ of 1 in Γ, where $N_{(\tau)}(t)=t\tau(t)\cdots\tau^{q-1}(t)$. If T has no proper idempotents then $T[X]/(X^q-t_0)$ has no proper idempotents.*

Proof. Suppose that T has no proper idempotents and $a^q=t_0$ for some a in T. Then $\tau(a^q)=(at)^q$ and hence $(\tau(a)a^{-1}t^{-1})^q=1$. Since X^q-1 is a separable polynomial, it follows that $\tau(a)a^{-1}t^{-1}=\gamma_i^i$ for some i (cf. [3, Lemma 2.1] and [5, Remark 2]). Then $\gamma_i^{-1}=N_{(\tau)}(\tau(a)a^{-1}t^{-1})=N_{(\tau)}(\gamma_i^i)=1$, which is a contradiction. Hence by Lemma 1.4, X^q-t_0 is irreducible in $T[X]$. Therefore $T[X]/(X^q-t_0)$ has no proper idempotents by Lemma 1.5.

Theorem 1.10. *Let q be a prime divisor of n, A a ring extension of B, σ a B-algebra automorphism of A of order q^2, and T the fixring of σ^q in A. Assume that A is a strongly cyclic $(\sigma^q; q)$-extension of T. If T has no proper idempotents, then so is A.*

Proof. By Th. 1.2, there exists an element $a\in U(A)$ such that $a^q\in U(T)$ and $\sigma^q(a)=\gamma_i a$ for some primitive q-th root γ_i of 1 in Γ. Set $t=a^{-1}\sigma(a)$ and $t_0=a^q$. Then $\sigma^q(t)=t\in U(T)$, $\sigma^q(t_0)=t_0\in U(T)$, $t\sigma(t)\cdots\sigma^{q-1}(t)=a^{-1}\sigma^q(a)=\gamma_i$, $t_0t^q=a^qa^{-q}(\sigma(a))^q=\sigma(t_0)$, and $\sigma|T$ is an automorphism of T of order q. Moreover, $T[X]/(X^q-t_0)$ is isomorphic to A by Th. 1.2. Hence, if T has no proper idempotents then, by Lemma 1.6, A has no proper idempotents.

Corollary 1.3. *Let m be a divisor of n, q a prime divisor of m, and $t>0$ an integer which is a multiple of q. Let A be a ring extension of B, σ a B-algebra automorphism of A of order tm, and T the fixring of σ^t in A. Assume A is a strongly cyclic $(\sigma^t; m)$-extension of T. If T has no proper idempotents then the fixring T_1 of σ^{tq} in A has no proper idempotents.*

Proof. By Th. 1.4, T_1 is a strongly cyclic $(\sigma^t|T_1, q)$-extension of T. Clearly $\sigma^{t/q}|T_1$ is an automorphism of T_1 whose order is a divisor of q^2. Since $(\sigma^{t/q}|T_1)^q=\sigma^t|T_1$ is of order q, it follows that $\sigma^{t/q}|T_1$ is of order q^2. Hence by Th. 1.10, the result is clear.

In what follows, let q_0 be the product of all different prime divisors of n

Theorem 1.11. *Let m be a divisor of n, and $t>0$ an integer which is a multiple of q_0. Let A be a ring extension of B, σ a B-algebra automorphism of A of order tm, and T the fixring τ^t in A. Assume A is a strongly cyclic $(\sigma^t; m)$-extension of T. If T has no proper idempotents,*

then so is A.

Proof. By making use of repeated use of Cor. 1.3, one will easily obtain the result.

Theorem 1.12. *Let A be a strongly cyclic* $(\sigma; n)$*-extension of B, and* B_1 *the fixring of* σ^{q_0} *in A. Then*

(1) *A has no proper idempotents if and only if* B_1 *has no proper idempotents.*

(2) *A is a field if and only if* B_1 *is a field.*

(3) *A is a domain if and only if* B_1 *is a domain.*

(4) *A is a local ring if and only if* B_1 *is a local ring.*

Proof. By Th. 1.4, A is a strongly cyclic $(\sigma^{q_0}; n/q_0)$-extension of B_1. Hence by Th. 1.11, we obtain (1) and (2). Moreover, (1) enables us to apply the same argument as in the proof of [6, Th. 1.8] to obtain (3) and (4).

Combining Th. 1.12 with Th. 1.5, we have

Corollary 1.4. *Let A be a strongly cyclic* $(\sigma; n)$*-extension of B, and* B_1 *as in Th.* 1.12. *Let C be a B-algebra with an identity element, N a proper ideal of B, and S a multiplicatively closed subset of B not containing* 0. *Then*

(1) $C \otimes {}_BA$ (*resp.* A/AN (*resp.* $A[S^{-1}]$)) *has no proper idempotents if and only if* $C \otimes {}_BB_1$ (*resp.* B_1/B_1N (*resp.* $B_1[S^{-1}]$)) *has no proper idempotents.*

(2) $C \otimes {}_BA$ (*resp.* A/AN (*resp.* $A[S^{-1}]$)) *is a field if and only if* $C \otimes {}_BB_1$ (*resp.* B_1/B_1N (*resp.* $B_1[S^{-1}]$)) *is a field.*

(3) $C \otimes {}_BA$ (*resp.* A/AN (*resp.* $A[S^{-1}]$)) *is a domain if and only if* $C \otimes {}_BB_1$ (*resp.* B_1/B_1N (*resp.* $B_1[S^{-1}]$)) *is a domain.*

(4) $C \otimes {}_BA$ (*resp.* A/AN (*resp.* $A[S^{-1}]$)) *is a local ring if and only if* $C \otimes {}_BB_1$ (*resp.* B_1/B_1N (*resp.* $B_1[S^{-1}]$)) *is a local ring.*

Theorem 1.13. *Let* A *be a ring extension of B,* σ *a B-algebra automorphism of A of order* n^s $(s > 0)$, *and* T_i *the fixring of* σ^{n^i} *in A* $(s \geqq i \geqq 0)$. *Assume that for every* $i > 0$, T_i *is a strongly cyclic* $(\sigma^{n^{i-1}} | T_i; n)$*-extension of* T_{i-1}. *Then*

(1) *A has no proper idempotents if and only if* T_1 *has no proper idempotents.*

(2) *A is a field if and only if T_1 is a field.*
(3) *A is a domain if and only if T_1 is a domain.*
(4) *A is a local ring if and only if T_1 is a local ring.*

Proof. Applying the result of Th. 1.11, the proof proceeds in the same way as in the proof of Th. 1.12.

2. Strongly abelian extensions. If a field A is a Galois extension of B with the Galois group $(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$ where the order of (σ_i) is a divisor of n, then it is seen that the fixring of $(\sigma_1)\times\cdots\times(\sigma_{i-1})\times(\sigma_{i+1})\times\cdots\times(\sigma_e)$ in A is a strongly cyclic extension of B. We shall introduce here the following definition.

Definition 2.1. Let A/B be a Galois extension with a Galois group $(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_e)$ and $\mathfrak{G}_i=(\sigma_1)\times\cdots\times(\sigma_{i-1})\times(\sigma_{i+1})\times\cdots\times(\sigma_e)$. If the fixring A_i of $\mathfrak{G}_i$ in A is a strongly cyclic $(\sigma_i|A_i;\ n_i)$-extension of B, then A/B will be called *a strongly abelian $(\sigma_1, \cdots, \sigma_e;\ n_1, \cdots, n_e)$-extension.*

It is easy to prove the following theorems which correspond Th. 1.6 and Th. 1.7 respectively.

Theorem 2.1. *Let A_i $(1\leqq i\leqq e)$ be strongly cyclic $(\sigma_i;\ n_i)$-extensions of B. Then $A_1\otimes_B A_2\otimes\cdots\otimes_B A_e$ is a strongly abelian $(\sigma_1, \cdots, \sigma_e;\ n_1, \cdots, n_e)$-extension of B.*

Theorem 2.2. *Let A be a strongly abelian $(\sigma_1, \cdots, \sigma_e;\ n_1, \cdots, n_e)$-extension of B. If A_i is the fixring of $(\sigma_1)\times\cdots\times(\sigma_{i-1})\times(\sigma_{i+1})\times\cdots\times(\sigma_e)$ in A, then A is isomorphic to $A_1\otimes_B A_2\otimes\cdots\otimes_B A_e$.*

Next, corresponding to Th. 1.12, we shall present the following

Theorem 2.3. *Let A be a strongly abelian $(\sigma_1, \cdots, \sigma_e;\ n_1, \cdots, n_e)$-extension of B. Let R_1 be the fixring of $(\sigma_1^{k_1})\times\cdots\times(\sigma_e^{k_e})$ in A, where k_i is the product of all different prime divisors of n_i.*

(1) *A has no proper idempotents if and only if R_1 has no proper idempotents.*
(2) *A is a field if and only if R_1 is a field.*
(3) *A is a domain if and only if R_1 is a domain.*
(4) *A is a local ring if and if R_1 is a local ring.*

By Theorems 1.3, 2.1, 2.2 and 2.3, we have the following

Theorem 2.4. *Let n_i $(1 \leqq i \leqq e)$ be divisors of n and k_i the product of all different prime divisors of n_i. Then*

(1) *there is a strongly abelian $(n_1, \cdots, n_e)$-extension of B which has no proper idempotents if and only if there is a strongly abelian $(k_1, \cdots, k_e)$-extension of B which has no proper idempotents.*

(2) *There is a strongly abelian $(n_1, \cdots, n_e)$-extension of B which is a domain if and only if there is a strongly abelian $(k_1, \cdots, k_e)$-extension of B which is a domain.*

(3) *There is a strongly abelian $(n_1, \cdots, n_e)$-extension of B which is a local ring if and only if there is a strongly abelian $(k_1, \cdots, k_e)$-extension of B which is a local ring.*

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