ON CYCLIC EXTENSIONS OF COMMUTATIVE RINGS

Dedicated to Professor Takeshi Inagaki on his 60th birthday

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In [3], K. Kishimoto presented a theory of abelian extensions of rings which contains a theory of cyclic extensions of commutative rings without proper idempotents. In this paper, we give a sharpening of Kishimoto's theory for commutative rings, and we also generalize some classical theorems in the theory of abelian extensions of fields to commutative rings. In §1, we shall give a theory of cyclic extensions of commutative rings. §2 is devoted to studying abelian extensions as an application of the theory of §1.

In all that follows B will mean a commutative algebra over the prime field $GF(p)$ ($p\neq 0$), and all ring extensions of B will be assumed to be commutative and have identities coinciding with the identity of B. As to other terminologies used in this paper, we follow [1] and [2].

The following lemma is useful in our paper

Lemma 0. *Let $\mathfrak{G}$ be a finite cyclic group of automorphisms in an arbitrary ring A which is generated by σ. If there exist elements a and b in A such that $t_{\mathfrak{G}}(a)$ $(=\sum_{\tau\in\mathfrak{G}}\tau(a))=1$ and $t_{\mathfrak{G}}(b)=0$ then there exists an element c in A such that $\sigma(c)=c+b$.*

In fact, if we correspond $\sigma\to b$, $\sigma^2\to b+\sigma(b)$, $\cdots$, $\sigma^n\to b+\sigma(b)+\cdots+\sigma^{n-1}(b)$ where n is the order of $\mathfrak{G}$, then by [5, §10, p. 65], we have the requested result.

1. Cyclic extensions of commutative rings. A ring extension A of B will be called a cyclic p^m-extension of B (with a Galois group (σ)) if A is a Galois extension of B with a cyclic Galois group (σ) of order p^m.

Lemma 1.1. *Let $f(X)=X^p-X-b_0\in B[X]$. Then $f(X)$ is separable. If there is a ring extension A of B which contains an element a such that $f(a)=0$ then $f(X)=(X-a)(X-(a+1))\cdots(X-(a+p-1))$; and if, in addition, there exists a B-algebra automorphism σ such that $\sigma(a)=a+1$*

then there exists an isomorphism

$$\text{(i)} \qquad B[X]/(f(X)) \rightarrow B[a]$$

such that $g(X)+(f(X)) \rightarrow g(a)$.

Proof. The first assertion is the result of [4, Cor. 4]. The second assertion is proved by making use of the same method as in the proof of [4, Lemma].

Theorem 1.1. *Let* $f(X)=X^p-X-b_0\in B[X]$. *Then* $B[X]/(f(X))$ *is a cyclic* p*-extension of* B *with a Galois group generated by an automorphism*

$$\text{(ii)} \qquad B[X]/(f(X)) \rightarrow B[X]/(f(X))$$

such that $X+(f(X)) \rightarrow X+1+(f(X))$.

Proof. The map $\sum_i b_iX^i \rightarrow \sum_i b_i(X+1)^i$ defines an automorphism of $B[X]$ sending $f(X)$ into $f(X+1)=f(X)$. Hence this induces an automorphism σ of $B[X]/(f(X))$. Clearly the cyclic group (σ) generated by σ is of order p. Set $x=X+(f(X))$ and let B' be the fixring of (σ) in $B[x]$. Then, for $c'\in B'$, we may write $c'=\sum_{i=0}^{p-1} c_ix^i$ where the c_i are elements of B. Hence we have $\sum_{i=1}^{p-1} c_i(\sigma^j(x))^i+(c_0-c')(\sigma^j(x))^0=0$ $(0\leqq j<p)$. The determinant of the matrix $\|(\sigma^j(x))^i\|$ $(0\leqq i, j<p)$ is $\pm\prod_{j<k}(\sigma^j(x)-\sigma^k(x))$ which is inversible in $B[x]$. This implies $c_0-c'=0$, that is, $c_0=c'$. Thus we obtain $B'=B$. Therefore, by [4, Lemma], $B[x]$ is a Galois extension of B with a Galois group (σ).

The following corollary is a direct consequence of Lemma 1.1 and Th. 1.1.

Corollary 1.1. *Let* A *be a ring extension of* B, *and* σ *a* B*-algebra automorphism of* A. *If there exists an element* a *of* A *such that* $a^p-a \in B$ *and* $\sigma(a)=a+1$. *Then there exist isomorphisms*

$$\begin{array}{ccc} B[X]/(f(X)) & \xrightarrow{\text{(i)}} & B[a] \\ \text{(ii)}\downarrow & & \downarrow \sigma|B[a] \\ B[X]/(f(X)) & \xrightarrow{\text{(i)}} & B[a] \end{array}$$

where $f(X)=X^p-X-(a^p-a)$ *and* $\sigma|B[a]$ *is the restriction of* σ *to* $B[a]$;

hence $B[a]$ *is a cyclic p-extension of* B *with a Galois group* $(\sigma|B[a])$.

Theorem 1.2. *Let* A *be a cyclic p-extension of* B *with a Galois group* (σ). *Then there exists an element* a *in* A *such that* $\sigma(a)=a+1$. *In this case, there holds that* $a^p-a\in B$ *and* $B[a]=A$; *and there exists an isomorphism*

$$\text{(i)} \qquad B[X]/(f(X)) \rightarrow A$$

where $f(X)=X^p-X-(a^p-a)$.

Proof. By Lemma 0 and [1, Lemma 1.6], there exists an element a in A such that $\sigma(a)=a+1$. Then $a^p-a\in B$. Hence by Cor. 1.1, $B[a]$ is a cyclic p-extension of B with a Galois group $(\sigma|B[a])$; whence there exist elements $x_1, \cdots, x_n$; $y_1, \cdots, y_n$ of $B[a]$ such that $\sum_{i=1}^{n} x_i\sigma^{j-1}(y_i)=\delta_{1,j}$ for $1\leq j\leq p$. Then for every $u\in A$, we have $u=\sum_{i=1}^{n} x_i\, t_{(\sigma)}(uy_i)\in B[a]$. This implies $B[a]=A$.

As a corollary to Th. 1.2, we have the following

Corollary 1.2. *Let* A *and* A' *be cyclic p-extensions of* B *with Galois groups* (σ) *and* (σ') *respectively which contain elements* a *and* a' *respectively such that* $\sigma(a)=a+1$, $\sigma'(a')=a'+1$ and $a^p-a=a'^p-a'$. *Then* $A=B[a]\cong A'=B[a'](a\longleftrightarrow a')$ *as B-algebras.*

Theorem 1.3. *Let* A *be a cyclic* p^e*-extension of* B *with a Galois group* (σ). *Then there exist elements* a, a_0 *in* A *such that*

$$t_{(\sigma)}(a)=1, \text{ and } \sigma(a_0)-a_0=a^p-a.$$

In this case, $A[X]/(X^p-X-a_0)$ *is a cyclic* p^{e+1}*-extension of* B *with a Galois group generated by an automorphism*

$$\text{(iii)} \qquad A[X]/(X^p-X-a_0) \rightarrow A[X]/(X^p-X-a_0)$$

such that $\sum_i c_iX^i+(X^p-X-a_0) \rightarrow \sum_i\sigma(c_i)(X+a)^i+(X^p-X-a_0)$, *which is an extension of* σ.

Proof. The first assertion is a direct consequence of [1, Lemma 1.6] and Lemma 0. It will be easily seen that the map $\sum_i c_iX^i \rightarrow \sum_i\sigma(c_i)(X+a)^i$ defines an automorphism of $A[X]$ sending X^p-X-a_0 into $(X+a)^p-(X+a)-a_0=X^p-X-a_0$. Hence this induces an automorphism

σ_0 of $A[X]/(X^p-X-a_0)$. We set $A[x]=A[X]/(X^p-X-a_0)$ where $x=X+(X^p-X-a_0)$. Then A $(\subset A[x])$ is a cyclic p^e-extension of B with a Galois group $(\sigma_0|A)$. Since $\sigma_0^{p^e}|A$ is an identity and $\sigma_0^{p^e}(x)=x+t_{(\sigma)}(a)=x+1$, it follows from Cor. 1.1 that $A[x]$ is a cyclic p-extension of A with a Galois group $(\sigma_0^{p^e})$. Hence there exist elements $u_1, \cdots, u_m$; $v_1, \cdots, v_m$ of A and elements $x_1, \cdots, x_n$; $y_1, \cdots, y_n$ of $A[x]$ such that $\sum_i u_i\rho(v_i)=\delta_{1,\rho}$ for $\rho\in(\sigma_0|A)$ and $\sum_j x_j\eta(y_j)=\delta_{1,\eta}$ for $\eta\in(\sigma_0^{p^e})$. Then $\sum_{i,j} x_j u_i\pi(v_i y_j)=\delta_{1,\pi}$ for $\pi\in(\sigma_0)$. Therefore $A[x]$ is a cyclic p^{e+1}-extension of B with a Galois group (σ_0).

Theorem 1.4. *Let A_* be a cyclic p^{e+1}-extension of B with a Galois group (σ_*). Let A be the fixring of $(\sigma_*^{p^e})$ in A_*. Then A is a cyclic p^e-extension of B with a Galois group (σ) generated by $\sigma=\sigma_*|A$ and there exist elements a, a_0 in A and an A-algebra isomorphism* (i′) *such that*

$$t_{(\sigma)}(a)=1, \qquad \sigma(a_0)-a_0=a^p-a,$$

and the following diagram

$$\begin{array}{ccc} A[X]/(X^p-X-a_0) & \xrightarrow{(\mathrm{i}')} & A_* \\ (\mathrm{iii})\ \downarrow & (\mathrm{i}') & \downarrow \sigma_* \\ A[X]/(X^p-X-a_0) & \longrightarrow & A_* \end{array}$$

is commutative where (i′) *is defined as for $B[X]/(f(X))$ in Lemma* 1.1, *and* (iii) *is given by*

$$\textstyle\sum_i c_iX^i+(X^p-X-a_0) \rightarrow \sum_i\sigma_*(c_i)(X+a)^i+(X^p-X-a_0).$$

Proof. It is obvious that A is a cyclic p^e-extension of B with a Galois group (σ) generated by $\sigma=\sigma_*|A$. Hence there exists an element a of A such that $t_{(\sigma)}(a)=1$. Noting $t_{(\sigma_*)}(a)=0$, there exists an element x of A_* such that $\sigma_*(x)=x+a$. Then $\sigma_*^{p^e}(x)=x+t_{(\sigma)}(a)=x+1$. We set $a_0=x^p-x$. Then $a_0\in A$ and $\sigma_*(a_0)-a_0=(x+a)^p-(x+a)-a_0=a^p-a$. Since $\sigma_*^{p^e}(x)=x+1$, by Th. 1.2, we have an A-algebra isomorphism

(i′) $$A[X]/(X^p-X-a_0) \rightarrow A[x]=A_*$$

such that $X+(X^p-X-a_0)\rightarrow x$. Then the diagram as in the theorem is commutative.

Theorem 1.5. *Let A be a cyclic p^e-extension of B with a Galois group (σ). Then*

(1) *if C is a B-algebra with an identity element then $C \otimes_B A$ is a cyclic p^e-extension of C (identifying $C \otimes 1$) with a Galois group $(1 \otimes \sigma)$.*

(2) *If N is a proper ideal of B then $AN \cap B = N$ and A/AN ($\cong B/N \otimes_B A$) is a cyclic p^e-extension of B/N (with a Galois group $(1 \otimes \sigma)$).*

(3) *If S is a multiplicatively closed subset of B containing 1 but not containing 0 then the quotient ring $A[S^{-1}]$ ($\cong B[S^{-1}] \otimes_B A$) is a cyclic p^e-extension of $B[S^{-1}]$ (with a Galois group $(1 \otimes \sigma)$).*

Proof. (1) Since B is a direct summand of a B-module A, we have $C \cong C \otimes B \subset C \otimes_B A$. Clearly $1 \otimes \sigma$ is a $C \otimes B$-algebra automorphism of $C \otimes_B A$ such that $(1 \otimes \sigma)^{p^e}$ is an identity. Let B_i be the fixring of (σ^{p^i}) in A. Then $B = B_0 \subset B_1 \subset \cdots \subset B_e = A$ and for every $i < e$, B_{i+1} is a cyclic p-extension of B_i with a Galois group $(\sigma^{p^i} | B_{i+1})$; hence by Th. 1.2, there exists an element u in B_{i+1} such that $B_i[u] = B_{i+1}$, $u^p - u \in B_i$, and $\sigma^{p^i}(u) = u + 1$. Then $(C \otimes B_i)[1 \otimes u] = C \otimes B_{i+1}$, $(1 \otimes u)^p - 1 \otimes u \in C \otimes B_i$, and $(1 \otimes \sigma)^{p^i}(1 \otimes u) = 1 \otimes u + 1 \otimes 1$. Hence by Cor. 1.1, $C \otimes B_{i+1}$ is a cyclic p-extension of $C \otimes B_i$ with a Galois group $((1 \otimes \sigma)^{p^i} | C \otimes B_{i+1})$. Therefore $C \otimes_B A$ is a cyclic p^e-extension of $C \otimes B$ with a Galois group $(1 \otimes \sigma)$. Thus we obtain (1). As corollaries to (1), we have (2) and (3).

Let B be an arbitrary ring with an identity. $f(X) \in B[X]$ will be called irreducible in $B[X]$ if each proper factor of $f(X)$ is contained in B. For an irreducible polynomial, we have the following

Lemma 1.2. *Let B be a ring without proper idempotents, and $f(X) = X^p - X - b_0 \in B[X]$. Then, $f(X)$ is irreducible if and only if $f(b) \neq 0$ for each $b \in B$.*

Proof. Let $f(b) \neq 0$ for each $b \in B$. Set $f(X) = g(X)h(X)$ where $g(X)$ is a monic polynomial in $B[X]$ of positive degree. Then by Lemma 1.1, $f(X)$ is separable and so $g(X)$ is separable. Hence by [2, Th. 2.2], there is a Galois extension A of B without proper idempotents which contains an element a such that $f(a) = 0$. From $f(a) = 0$, we have $f(X) = (X - a)(X - (a+1)) \cdots (X - (a+p-1))$. Since $a \notin B$, there exists a B-algebra automorphism σ of A such that $\sigma(a) \neq a$. Then, by [2, Lemma 2.1], we have $\sigma(a) = a + m$, $0 < m < p$. On the other hand, let C be a splitting ring of

$g(X)$ over A without proper idempotents. Then there exists an element $c\in C$ with $g(c)=0$. Since $f(c)=0$, we have $c=a+i=\sigma^k(a)$ for some i $(0\leqq i<p)$ and k. Therefore we obtain deg $g(X)\geqq p$; this implies $g(X)=f(X)$. Thus $f(X)$ is irreducible. The converse is obvious.

Theorem 1.6. *Let B be a ring without proper idempotents, and $f(X)=X^p-X-b_0\in B[X]$. Then, $B[X]/(f(X))$ has no proper idempotents if and only if $f(b)\neq 0$ for each $b\in B$.*

Proof. By Lemma 1.1, $f(X)$ is separable. Hence, $B[X]/(f(X))$ has no proper idempotents if and only if $f(X)$ is irreducible. By Lemma 1.2, this is equivalent to that $f(b)\neq 0$ for each $b\in B$.

As a direct consequence of Th. 1.1 and Th. 1.6, we have the following theorem which contains the result of [3, Th. 3.1].

Theorem 1.7. *Let B be a ring without proper idempotents. Then, there is a cyclic p-extension of B without proper idempotents if and only if there is an element b_0 of B such that $b^p-b-b_0\neq 0$ for each $b\in B$.*

Corollary 1.3. *Let A be a cyclic p-extension of B without proper idempotents. If a is an element of A such that $a^p-a\in B$ and $a\notin B$ then $A=B[a]\cong B[X]/(X^p-X-b_0)$ where $b_0=a^p-a$.*

Proof. Let (σ) be the Galois group of the cyclic p-extension A of B, and a an element of A such that $a^p-a=b_0\in B$ and $a\notin B$. Then $\sigma(a)\neq a$. Since $f(X)=X^p-X-b_0$ is separable and $f(\sigma(a))=0$, it follows from [2, Lemma 2.1] that $\sigma(a)\in\{a, a+1, \cdots, a+p-1\}$; hence $\sigma(a)=a+m$, $0<m<p$. Hence $\sigma^i(a)=a+1$ for some i. Then by Th. 1.2, we have $B[X]/(f(X))\cong B[a]=A$.

From Cor. 1.3, we have the following

Corollary 1.4. *Let A_1, A_2 be cyclic p-extensions of B without proper idempotents. Then, $A_1\cong A_2$ as B-algebras if and only if there exist elements $a_1\in A_1$ and $a_2\in A_2$ such that $a_1^p-a_1=a_2^p-a_2\in B$ and $a_1\notin B$.*

Lemma 1.3. *Let A be a cyclic p^e-extension of B with a Galois group (σ) which has no proper idempotents. If a, a_0 are elements in A such that $\sigma(a_0)-a_0=a^p-a$ and $t_{(\sigma)}(a)=1$ then $A[X]/(X^p-X-a_0)$ has no proper idempotents.*

Proof. Suppose that $t^p - t - a_0 = 0$ for some $t \in A$. Then $a^p - a = \sigma(a_0) - a_0 = (\sigma(t) - t)^p - (\sigma(t) - t)$. Hence $(\sigma(t) - t - a)^p - (\sigma(t) - t - a) = 0$. Since the polynomial $X^p - X$ is separable, it follows that $\sigma(t) - t - a$ is an element of $GF(p)$; then $t_{(\sigma)}(\sigma(t) - t - a) = 0$. On the other hand, we have $t_{(\sigma)}(\sigma(t) - t - a) = t_{(\sigma)}(-a) = -1$. This is a contradiction. Hence by Th. 1.6, we obtain our assertion.

The result of the following lemma is well known.

Lemma 1.4. *Let A be a Galois extension of B. Let M be a maximal ideal of B. Then*

(1) *there exist maximal ideals $M_1, \cdots, M_n$ of A such that $M_1 \cap \cdots \cap M_n = AM$.*

(2) *If A is a local ring with a maximal ideal M_* then B is a local ring with a maximal ideal $B \cap M_*$ and $M_* = A(B \cap M_*)$.*

(3) *When B is a local ring with a maximal ideal M, A is a local ring if and only if AM is a maximal ideal of A.*

(4) *B is a semi-local ring if and only if so is A.*

Theorem 1.8. *Let A be a cyclic p^e-extension of B with a Galois group (σ), and B_1 the fixring of (σ^p) in A. Then*

(1) *A has no proper idempotents if and only if B_1 has no proper idempotents.*

(2) *A is a field if and only if B_1 is a field.*

(3) *A is a domain if and only if B_1 is a domain.*

(4) *A is a local ring if and only if B_1 is a local ring.*

Proof. For $0 \leqq i \leqq e$, let B_i be the fixring of (σ^{p^i}) in A. Then $B \subset B_1 \subset B_2 \cdots \subset B_e = A$, and for $j > i$, B_j is a cyclic p^{j-i}-extension of B_i with a Galois group $(\sigma^{p^i} | B_j)$. Hence (1) is a direct consequence of Th. 1.4 and Lemma 1.3. (2) follows from (1). (3) Let B_1 be a domain. Let Q_1 be the quotient field of B_1 and Q the quotient field of B in Q_1. Then $Q[B_1]$ is a finitely generated Q-module. This implies $Q_1 = Q[B_1] \cong Q \otimes_B B_1$. By Th.1.5, $Q \otimes_B A$ is a cyclic p^e-extension of $Q \otimes B$. Since the canonical homomorphism $Q \otimes_B B_1 \to Q \otimes_B A$ is injective, it follows from (2) that $Q \otimes_B A$ is a field. Noting $A \cong 1 \otimes A \subset Q \otimes_B A$, A is a domain. (4) If A is a local ring then, by Lemma 1.4, B_1 is a local ring. To see the converse, let B_1 be a local ring with a maximal ideal M_1, and set $M = B \cap M_1$. Then by Lemma 1.4, we have $M_1 = B_1 M$. By Th. 1.5, A/AM is a cyclic p^e-extension of

B/M. Since $B_1/(AM \cap B_1) = B_1/M_1$ is a field, it follows from (2) that A/AM is a field, that is, AM is a maximal ideal of A. Therefore by Lemma 1.4, A is a local ring.

The following corollary is a direct consequence of Th. 1.5 and Th. 1.8.

Corollary 1.5. *Let A be a cyclic p^e-extension of B with a Galois group (σ), and B_1 the fixring of (σ^p) in A. Let C be a B-algebra with an identity element, N a proper ideal of B, and S a multiplicatively closed subset of B not containing* 0. *Then*

(1) *$C\otimes_B A$ (resp. A/AN (resp. $A[S^{-1}]$)) has no proper idempotents if and only if $C\otimes_B B_1$ (resp. B_1/B_1N (resp. $B_1[S^{-1}]$)) contains no proper idempotents.*

(2) *$C\otimes_B A$ (resp. A/AN (resp. $A[S^{-1}]$)) is a field if and only if $C\otimes_B B_1$ (resp. B_1/B_1N (resp. $B_1[S^{-1}]$)) is a field.*

(3) *$C\otimes_B A$ (resp. A/AN (resp. $A[S^{-1}]$)) is a domain if and only if $C\otimes_B B_1$ (resp. B_1/B_1N (resp. $B_1[S^{-1}]$)) is a domain.*

(4) *$C\otimes_B A$ (resp. A/AN (resp. $A[S^{-1}]$)) is a local ring if and only if $C\otimes_B B_1$ (resp. B_1/B_1N (resp. $B_1[S^{-1}]$)) is a local ring.*

The following theorem contains the results of [3, Th. 3.2, Th. 5.2]. K. Kishimoto proved (2) and (3) directly. However, the corollary is a direct consequence of Th. 1.3 and Th. 1.8.

Corollary 1.6. *Let e be a positive integer. Then*

(1) *there is a cyclic p^e-extension of B which has no proper idempotents if and only if there is a cyclic p-extension of B which has no proper idempotents.*

(2) (K. Kishimoto). *There is a cyclic p^e-extension of B which is a domain if and only if there is a cyclic p-extension of B which is a domain.*

(3) (K. Kishimoto). *There is a cyclic p^e-extension of B which is a local ring if and only if there is a cyclic p-extension which is a local ring.*

2. Abelian extensions of commutative rings. If A is an abelian extension of B with a Galois group $(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_n)$ where for i, (σ_i) is of order p^{e_i} then, it is called to be an abelian $(p^{e_1}, \cdots, p^{e_n})$-extension of B (with a Galois group $(\sigma_1)\times(\sigma_2)\times\cdots\times(\sigma_n)$).

The following theorem will be proved easily.

Theorem 2.1. *Let A_i $(1\leqq i\leqq n)$ be cyclic p^{e_i}-extension of B with Galois groups (σ_i). Then $A_1\otimes_B A_2\otimes\cdots\otimes_B A_n$ is an abelian $(p^{e_1},\cdots,p^{e_n})$-extension of B with a Galois group $(\sigma_1)\times\cdots\times(\sigma_n)$.*

The following theorem follows from [1, Th. 2.3] and the fact that every cyclic p^e-extension of B is a free B-module.

Theorem 2.2. *Let A be an abelian $(p^{e_1},\cdots,p^{e_n})$-extension with a Galois group $(\sigma_1)\times\cdots\times(\sigma_n)$. For each i, let A_i be the fixring of $(\sigma_1)\times\cdots\times(\sigma_{i-1})\times(\sigma_{i+1})\times\cdots\times(\sigma_n)$ in A. Then $A\cong A_1\otimes_B A_2\otimes\cdots\otimes_B A_n$, and the A_i are cyclic p^{e_i}-extensions with Galois groups $(\sigma_i|A_i)$.*

Now, we shall prove the following

Theorem 2.3. *Let A be an abelian $(p^{e_1},\cdots,p^{e_n})$-extension with a Galois group $(\sigma_1)\times\cdots\times(\sigma_n)$. Let B_1 be the fixring of $(\sigma_1^p)\times\cdots\times(\sigma_n^p)$ in A. Then*

(1) *A has no proper idempotents if and only if B_1 has no proper idempotents.*

(2) *A is a field if and only if B_1 is a field.*

(3) *A is a domain if and only if B_1 is a domain.*

(4) *A is a local ring if and only if B_1 is a local ring.*

Proof. For $1\leqq i\leqq n$ and for $1\leqq j\leqq e_i$, let B_{ij} be the fixring of $(\sigma_1)\times\cdots\times(\sigma_{i-1})\times(\sigma_i^{p^j})\times(\sigma_{i+1})\times\cdots\times(\sigma_n)$ in A. Then by Th. 2.2, we have $A\cong B_{1e_1}\otimes\cdots\otimes_B B_{ne_n}$, $B_1\cong B_{11}\otimes\cdots\otimes B_{n1}$, B_{ie_i} is a cyclic p^{e_i}-extension of B with a Galois group $(\sigma_i|B_{ie_i})$, and B_{i1} is the fixring of $(\sigma_i^p|B_{ie_i})$ in B_{ie_i}. Clearly $B_{1e_1}\otimes B_{21}\otimes\cdots\otimes B_{n1}$ is a cyclic p^{e_1}-extension of $B\otimes B_{21}\otimes\cdots\otimes B_{n1}$. Assume that $B_{11}\otimes B_{21}\otimes\cdots\otimes B_{n1}$ has no proper idempotents. Then by Th. 1.8, $B_{1e_1}\otimes B_{21}\otimes\cdots\otimes B_{n1}$ has no proper idempotents. By a similar method, we see that $B_{1e_1}\otimes B_{2e_1}\otimes B_{31}\otimes\cdots\otimes B_{n1}$ has no proper idempotents. Continuing this way, it follows that A has no proper idempotents. The converse is obvious. By similar methods, we have the other assertions.

By theorems 1.3, 2.1, 2.2, and 2.3, we have the following

Theorem 2.4. *Let e_i $(1\leqq i\leqq n)$ be positive integers. Then*

(1) *there is an abelian $(p^{e_1},\cdots,p^{e_n})$-extension of B which has no proper idempotents if and only if there is an abelian $(\underbrace{p,\cdots,p}_{n})$-extension of B which has no proper idempotents.*

(2) *There is an abelian* $(p^{e_1}, \cdots, p^{e_n})$*-extension of B which is a domain if and only if there is an abelian* $(\underbrace{p, \cdots, p}_{n})$*-extension which is a domain.*

(3) *There is an abelian* $(p^{e_1}, \cdots, p^{e_n})$*-extension of B which is a local ring if and only if there is an abelian* $(\underbrace{p, \cdots, p}_{n})$*-extension which is a local ring.*

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