THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR nTH-ORDER DIFFERENTIAL EQUATION

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In this note we shall study the asymptotic behavior for $t \to \infty$ of solutions of the differential equation

(1) $$y^{(n)} + f(t, y) = 0$$

and prove:

Theorem. *If $f(t, y)$ satisfies the conditions:*

i $f(t, y)$ *is continuous on* $D: (t \geqslant 0;\ -\infty < y < \infty)$,

ii $\dfrac{\partial^{n-1}}{\partial y^{n-1}} f(t, y) = A(t, y)$ *exists and* $A(t, 0) > 0$ *on* D,

iii $|f(t, y)| \leq A(t, 0)\ |y|$ *on* D, *and*

iv $\int^{\infty} t^{n-1} A(t, 0)\, dt < +\infty$, *then* (1) *has solutions which*

are asymptotic to $a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$, *where* $a_{n-1} \neq 0$.

One can find functions of t and y which satisfies the conditions given in the theorem. For example function $f(t, y) = \dfrac{y^{n-1} e^{-y^2}}{(1+t)^{n+1}}$ satisfies all the four conditions given in the theorem. Cohen [2] proved a similar theorem for the second order differential equation and for the case, $f(t, y) = \pm t^{\sigma} y^{\gamma}$, Bellman [1] has given an exhaustive treatment of the asymptotic behavior of solutions which exist and have continuous derivatives for $t \geqslant t_0$. There also exist several results on asymptotic behavior of solutions of second order equations with $f(t, y) = \alpha(t) y^{2n+1}$ and references can be found in the papers of Waltman [4] and Moore and Nehari [3].

Proof of the theorem.

We have

$$y^{(n)}(t) = -f(t, y).$$

1) The author acknowledges partial support of N. R. C. grant No. A5613,

Integrating n times between 1 and t, we obtain

$$(2) \qquad y(t)=C_0+C_1t+\cdots+C_{n-1}t^{n-1}-\frac{1}{(n-1)!}\int_1^t (t-s)^{n-1}f(s,y(s))ds.$$

From this we obtain, for $t\geqslant 1$

$$|y(t)|\leq(|C_0|+|C_1|+\cdots+|C_{n-1}|)t^{n-1} + \frac{1}{(n-1)!}t^{n-1}\int_1^t |f(s,y(s))|ds,$$

or

$$\frac{|y(t)|}{t^{n-1}}\leq|C_0|+|C_1|+\cdots+|C_{n-1}| + \frac{1}{(n-1)!}\int_1^t s^{n-1}A(s,0)\frac{|y(s)|}{s^{n-1}}ds.$$

Now using the Gronwall inequality ([1], p. 107, Lemma 1), we obtain

$$\frac{|y(t)|}{t^{n-1}}\leq(|C_0|+|C_1|+\cdots+|C_{n-1}|) \times\exp\left(\frac{1}{(n+1)!}\int_1^t s^{n-1}A(s,0)ds\right),$$

or

$$(3) \qquad \frac{|y(t)|}{t^{n-1}}\leq C, \text{ where } C \text{ is a positive constant.}$$

Differentiating (2), $(n-1)$ times with respect to t, we obtain

$$(4) \qquad y^{(n-1)}(t)=C_{n-1}-\int_0^t f(s,y(s))ds.$$

Again from (3) and $|f(t,y)|\leq A(t,0)\ |y(t)|$, we have

$$\int_1^t |f(s,y(s))|ds\leq\int_1^t A(s,0)\ |y(s)|ds \leq C\int_1^t s^{n-1}A(s,0)\ ds.$$

Thus the integral $\int_1^t |f(s,y(s))|ds$ converges as $t\to\infty$ and $y^{(n-1)}(t)$ has a limit as $t\to\infty$. To ensure that this limit is not equal to zero, we choose $C_{n-1}=1$ and a point t_0, where t_0 is chosen so that

$$1-C\int_{t_0}^{t} s^{n-1} A(s, 0)ds>0.$$

Now from $y^{(n-1)}(t)\sim$constant follows the theorem.

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(*Received June 15, 1971*)