ON THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS THE SUM OF A POWERFUL AND A SQUAREFREE INTEGERS

Dedicated to Professor TAKESHI INAGAKI on the occasion of his sixtieth birthday

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The problem of finding the number of representations of a positive integer as the sum of the square of an integer and a squarefree integer has been discussed by Th. Estermann [1 ; § 1]. A positive integer n is called squarefree, if n is not divisible by the square of any prime. He has shown that if $G(N)$ denotes the number of ways of representing a positive integer N as the sum of a square and a squarefree integer, then we have for any fixed $\varepsilon>0$

$$G(N)=C(N)N^{1/2}\prod_{p^2|N}\left(1-\frac{1}{p}\right)+O(N^{(1/3)+\varepsilon}),$$

where

$$C(N)=\prod_{p\nmid N}(1-\nu_p(N)\,p^{-2})$$

with

$$\nu_p(N)=\begin{cases}1+(-1)^{(N-1)/2} & (p=2)\\ 1+\left(\frac{N}{p}\right) & (p>2)\end{cases}$$

for primes p not dividing N, and where the constant implied in the symbol O depends only on ε. In particular, every sufficiently large positive integer can be represented as the sum of a square and a squarefree integer.

In the present paper we shall consider the number of representations of a positive integer as the sum of a powerful and a squarefree integers. A positive integer n is a powerful integer, by definition, if p^2 divides n whenever the prime p divides n (cf. [2]).

We shall denote by P the set of all powerful integers and by S the set of all squarefree integers. It is clear that the intersection $P \cap S$ consists of one element, the unity.

For every positive integer N we denote by $E(N)$ the number of ways of representing N in the form

$$N = a + b$$

with

$$a \in P, \quad b \in S \quad \text{and} \quad (a, b) = 1.$$

We shall prove the following

Theorem. *For any fixed positive number* ε *we have*

$$E(N) = K(N)\, N^{1/2} \prod_{p \mid N} \left(1 - \frac{1}{p}\right) + O(N^{(4/9)+\varepsilon}) \qquad (N > 0),$$

where

$$K(N) = \sum_{\substack{m=1 \\ (m, n)=1}}^{\infty} \frac{\mu^2(m)}{m^{3/2}} C(mN)$$

and the O-constant may depend only on ε.

In particular, every sufficiently large positive integer N admits a representation of the form

$$N = a + b$$

with

$$1 < a \in P, \quad 1 < b \in S \text{ and } (a, b) = 1,$$

since we have

$$K(N) \geqq \prod_{p} \left(1 - \frac{2}{p^2}\right) > 0$$

and

$$\prod_{p \mid N} \left(1 - \frac{1}{p}\right) > \frac{c}{\log \log 3N}$$

for some absolute constant $c > 0$.

1. Lemmata. In order to establish the theorem stated above, we require some auxiliary results.

The letters d, k, m, n, N denote positive integers and p a prime number, and ε is used to denote an arbitrarily small but fixed positive number. As usual, $\mu(n)$ is the Möbius function, $\varphi(n)$ is the Euler totient function and $\tau(n)$ is the divisor function giving the number of positive divisors of n. Also, $v(n)$ denotes the number of distinct prime divisors of n.

Lemma 1. *Let $d \in S$ and suppose that n is a quadratic residue* (mod d^2). *Then, the number $s(d)$ of the incongruent solutions z of the congruence*

$$z^2 \equiv n \pmod{d^2}$$

is given by

$$s(d) = 2^{v(d)}.$$

For a proof of this lemma one may refer e. g. to [3; Theorem 47]. We note that the integer n is a quadratic residue (mod d^2) if and only if n is a quadratic residue (mod p) for all prime factors p of d, and further $n \equiv 1$ (mod 4) when d is even. Thus, if we write

$$e_d(n) = 2^{-v(n)} \prod_{p|d} \nu_p(n),$$

then $e_d(n) = 1$ or 0 according as n is or is not a quadratic residue (mod d^2).

Lemma 2. *Let a and b be positive integers and let $Q(N;\ a, b)$ denote the number of pairs of positive integers m, n satisfying*

$$am^2 + bn^2 = N.$$

Then, we have

$$Q(N;\ a, b) \leqq 2\tau(N).$$

This is [1; Hilfssatz 1].

2. Proof of the Theorem. Using the relation

$$\mu^2(n) = \sum_{d^2|n} \mu(d)$$

and noticing that $(a, b) = 1$ is equivalent to $(a, N) = 1$ when $N = a + b$, we have

$$E(N) = \sum_{\substack{a \leqq N \\ a \in P \\ (a, N) = 1}} \mu^2(N-a) = \sum_{\substack{a \leqq N \\ a \in P \\ (a, N) = 1}} \sum_{d^2 | N-a} \mu(d).$$

It is not difficult to see that every integer $a\in P$ can be uniquely written in the form

$$a=n^2 m^3$$

with $m\in S$ (cf. [2]). Hence, we may rewrite

$$E(N)=\sum_1\mu(d)+\sum_2\mu(d)+\sum_3\mu(d),$$

where $\sum_1$, $\sum_2$ and $\sum_3$ respectively indicate the summation over the positive integers d, m, n satisfying the conditions

$$\begin{cases} d\leqq t,\ m\leqq x,\ \mu(m)\neq 0,\ n^2m^3\leqq N, \\ \quad (mn, N)=1,\ d^2\,|\,N-n^2m^3, \end{cases}$$

$$\begin{cases} d\leqq N^{1/2},\ x<m\leqq N^{1/3},\ \mu(m)\neq 0,\ n^2m^3\leqq N, \\ \quad (mn, N)=1,\ d^2\,|\,N-n^2m^3 \end{cases}$$

and

$$\begin{cases} t<d\leqq N^{1/2},\ m\leqq x,\ \mu(m)\neq 0,\ n^2m^3\leqq N, \\ \quad (mn, N)=1,\ d^2\,|\,N-n^2m^3, \end{cases}$$

where t and x are fixed real numbers such that

$$1\leqq t\leqq N^{1/2} \quad\text{and}\quad 0<x\leqq N^{1/3}.$$

Firstly we have

$$\sum_1 \mu(d)=\sum_{\substack{d\leqq t\\(d,N)=1}} \mu(d)\, T_d,$$

where

$$\begin{aligned} T_d &= \sum_{\substack{m\leqq x\\(m,dN)=1}} \mu^2(m) \sum_{\substack{n\leqq (N/m^3)^{1/2}\\(n,N)=1\\ d^2|N-n^2m^3}} 1 \\ &= \sum_{\substack{m\leqq x\\(m,dN)=1}} \mu^2(m) \sum_{\substack{z^2\equiv mN(d^2)\\(z,N)=1\\0<z\leqq Nd^2}} \left(\left[\frac{1}{Nd^2}\left(\left(\frac{N}{m^3}\right)^{1/2}-z\right)\right]+1\right) \\ &= \sum_{\substack{m\leqq x\\(m,dN)=1}} \mu^2(m)e_d(mN)\frac{\varphi(N)s(d)}{Nd^2}\left(\frac{N}{m^3}\right)^{1/2}+O\Big(\sum_{m\leqq x} s(d)\Big) \\ &= N^{1/2}\,\frac{\varphi(N)}{N}\,\frac{s(d)}{d^2} \sum_{\substack{m=1\\(m,dN)=1}}^{\infty} \frac{\mu^2(m)e_d(mN)}{m^{3/2}} \end{aligned}$$

$$+O\left(\frac{s(d)}{d^2}N^{1/2}x^{-1/2}\right)+O(s(d)x).$$

Hence, noticing that

$$\sum_{d\leq t}\frac{s(d)}{d^2}=O(1) \text{ and } \sum_{d\leq t}s(d)=O(t\log 2t),$$

we find

$$\sum\nolimits_1\mu(d)=N^{1/2}\frac{\varphi(N)}{N}\sum_{\substack{d\leq t\\(d,N)=1}}\frac{\mu(d)s(d)}{d^2}\sum_{\substack{m=1\\(m,dN)=1}}^{\infty}\frac{\mu^2(m)e_d(mN)}{m^{3/2}}$$

$$+O\left(N^{1/2}x^{-1/2}\sum_{d\leq t}\frac{s(d)}{d^2}\right)+O(x\sum_{d\leq t}s(d))$$

$$=N^{1/2}\frac{\varphi(N)}{N}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu^2(m)}{m^{3/2}}\sum_{\substack{d=1\\(d,mN)=1}}^{\infty}\frac{\mu(d)s(d)e_d(mN)}{d^2}$$

$$+O\left(N^{1/2}\sum_{d>t}\frac{1}{d^2}\right)+O(N^{1/2}x^{-1/2})+O(xt\log 2t)$$

$$=N^{1/2}\frac{\varphi(N)}{N}\sum_{\substack{m=1\\(m,N)=1}}^{\infty}\frac{\mu^2(m)}{m^{3/2}}\prod_{p\nmid mN}\left(1-\frac{\nu_p(mN)}{p^2}\right)$$

$$+O(N^{1/2}t^{-1})+O(N^{1/2}x^{-1/2})+O(xt\log 2t),$$

by Lemma 1.

Next, we have

$$\sum\nolimits_2\mu(d)=\sum_{\substack{x<m\leq N^{1/2}\\(m,N)=1}}\mu^2(m)\sum_{\substack{n\leq(N/m^3)^{1/2}\\(n,N)=1}}\sum_{\substack{d\leq N^{1/2}\\d^2|N-n^2m^3}}\mu(d)$$

$$=O\left(\sum_{m>x}\sum_{n\leq(N/m^3)^{1/2}}\tau(N-n^2m^3)\right)$$

$$=O\left(N^{\varepsilon}\sum_{m>x}\sum_{n\leq(N/m^3)^{1/2}}1\right)$$

$$=O(N^{(1/2)+\varepsilon}x^{-1/2}),$$

since

$$\max_{n\leq N}\tau(n)=O(N^{\varepsilon}).$$

Finally, writing

$$kd^2+m^3n^2=N$$

for $d^2|N-n^2m^3$, we find

$$\sum_3 \mu(d)=O\Big(\sum_{k\leq Nt^{-2}}\sum_{m\leq x} Q(N;k,m^3)\Big)$$
$$=O(N^{1+\varepsilon}t^{-2}x)$$

by Lemma 2.

We thus obtain

$$E(N)=K(N)\,N^{1/2}\frac{\varphi(N)}{N}+R(N),$$

where

$$R(N)=O(N^{1/2}t^{-1})+O(xt\log 2t)$$
$$+O(N^{(1/2)+\varepsilon}x^{-1/2})+O(N^{1+\varepsilon}t^{-2}x)$$
$$=O(N^{(4/9)+\varepsilon}),$$

on taking

$$t=N^{1/3} \quad \text{and} \quad x=N^{1/9}.$$

This completes the proof of our theorem, since

$$\frac{\varphi(N)}{N}=\prod_{p|N}\Big(1-\frac{1}{p}\Big).$$

3. A Dual Problem. Estermann [1 ; § 2] has also given an asymptotic formula for the number $H(x)$ of squarefree integers not exceeding x and having the form n^2+l, where l is a given non-zero integer. Indeed, he has proved that

$$H(x)=C(-l)\,x^{1/2}\prod_{p^2|l}\Big(1-\frac{1}{p}\Big)+O(x^{1/3}\log 2x) \quad (x\geqq 1),$$

where the O-constant is dependent only on l.

An analogous formula can be found for the number $F(x)$ of positive integers $a\leqq x$ satisfying

$$a\in P,\ a+l\in S \quad \text{and} \quad (a,l)=1,$$

where l is again a fixed non-zero integer. By the method employed above, with [1 ; Hilfssatz 2] in place of Lemma 2, one may easily obtain for any positive number ε

$$F(x)=K(-l)x^{1/2}\prod_{p\mid l}\left(1-\frac{1}{p}\right)+O(x^{(4/9)+\varepsilon}) \qquad (x>0),$$

where the O-constant depends on l and ε.

A Final Remark. S. W. Golomb [2] gave an asymptotic formula for the number of powerful integers not exceeding a given limit, whereas much finer and more general results had already been obtained by P. T. Bateman and E. Crosswald: On a theorem of Erdös and Szekeres, Illinois J. Math., 2 (1958), 88—98.

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(*Received April 1, 1971*)