ON QUASI-FROBENIUS EXTENSIONS

Dedicated to Professor TAKESHI INAGAKI on the occasion of his sixtieth birthday

YOSHIMI KITAMURA

In his papers [9, 10], B. Müller introduced the notion of QF-extensions (quasi-Frobenius extensions) which is a generalization of that of Frobenius extensions defined by F. Kasch [6]. Recently, in [8] Y. Miyashita obtained a generalized endomorphism ring theorem for Frobenius extensions.

In this paper, the main theme of our discussion will concern mainly QF-extensions. In § **1**, we shall give a similar endomorphism ring theorem for QF-extensions (Theorem 1. 2) and a remark on symmetric extensions. In § **2**, we shall show that if a ring extension A/B is a right or left QF-extension then A is a right (resp. left) QF-3 ring if and only if B is a right (resp. left) QF-3 ring (Theorem 2. 4). Furthermore, one will find several additional results on QF-extensions.

The author extends his hearty thanks to Professor T. Onodera and Professor. Y. Miyashita for their helpful suggetion and encouragement.

0. Throughout the present paper, it is assumed that every ring has an identity, every module is unitary and that a subring of a ring contains the identity of the ring. When B is a subring of a ring A, we shall say that A/B is a ring extension. Now, let A and A' be rings, and consider A-A'-modules M and N. By M^n we denote the direct sum of n copies of M. If M is isomorphic to a direct summand of N as A-A'-module, we shall write ${}_AM_{A'} < \oplus\, {}_AN_{A'}$. If ${}_AM_{A'} < \oplus\, {}_AN^n_{A'}$ for some n, we shall write ${}_AM_{A'} \mid {}_AN_{A'}$, and if ${}_AM_{A'} \mid {}_AN_{A'}$ and ${}_AN_{A'} \mid {}_AM_{A'}$ then we shall write ${}_AM_{A'} \sim {}_AN_{A'}$. Finally, if M is isomorphic to a submodule of N as A-A'-module, we shall write ${}_AM_{A'} \subsetneq {}_AN_{A'}$.

A ring extension A/B is called a *Frobenius* (resp. *right* QF-) *extension* if the following conditions are satisfied :

(1) $A_B \mid B_B$ and (2) ${}_BA_A \cong {}_B\mathrm{Hom}(A_B, B_B)_A$ (resp. ${}_BA_A \mid {}_B\mathrm{Hom}(A_B, B_B)_A$).

A *left* QF-*extension* is defined similarly. If A/B is a right and left QF-*extension* then it is called merely a QF-*extension*.

A ring R is said to be a *right* (resp. *left*) QF-3 *ring* if there exists a finitely generated, projective, injective and faithful right (resp. left) R-module. If for each proper left (resp. right) ideal of a ring R its right (resp. left) annihilator is non-zero, R is called a *right* (resp. *left*) *S-ring*. A ring R is said to be *right* (resp. *left*) *PF* if every faithful right (resp. *left*) R-module is completely faithful in the sense of Azumaya [1], namely, if $R_R \mid M_R$ (resp. ${}_RR \mid {}_RM$) for every faithful *right* (resp. *left*) R-module M. Finally, a *right* R-module M is said to be *torsionless* if for each non-zero x in M there exists some f in $\mathrm{Hom}(M_R, R_R)$ with $f(x)\neq 0$ (cf. [2]).

1. Let A/B be a ring extension, and M a right A-module. We shall put $A^*=\mathrm{End}(M_A)$ and $B^*=\mathrm{End}(M_B)$, which operate on the left side of M. Then there hold the following canonical isomorphisms :

$$ {}_{A^*}B^*{}_{B^*}\cong {}_{A^*}\mathrm{Hom}(M\otimes_B A_A,\ M_A)_{B^*},\quad \text{and} $$
$$ {}_{B^*}B^*{}_{A^*}\cong {}_{B^*}\mathrm{Hom}(M_A,\ \mathrm{Hom}(A_B,\ M_B)_A)_{A^*}. $$

Lemma 1. 1. *If* $M\otimes_B A_A \mid M_A$ *then* ${}_{A^*}B^* \mid {}_{A^*}A^*$.

Proof. ${}_{A^*}B^* \cong {}_{A^*}\mathrm{Hom}(M\otimes_B A_A,\ M_A) \mid {}_{A^*}\mathrm{Hom}(M_A,\ M_A)$.

Theorem 1 .2. *Assume* $M\otimes_B A_A \mid M_A$. *If* A/B *is a left* QF-*(resp.* QF-*) extension then* B^*/A^* *is a left* QF-*(resp.* QF-*) extension.*

Proof. If A/B is a left QF-(resp. QF-) extension then ${}_B\mathrm{Hom}(A_B,\ B_B)_A \mid$ (resp. $\sim$)${}_BA_A$. By $A_B \mid B_B$, we have ${}_{B^*}\mathrm{Hom}(A_B,\ M_B)_A \cong {}_{B^*}M\otimes_B\mathrm{Hom}(A_B,\ B_B,\ )_A$ canonically, and so ${}_{B^*}\mathrm{Hom}(A_B,\ M_B)_A \mid$ (resp. $\sim$)${}_{B^*}M\otimes_B A_A$. In particular, $\mathrm{Hom}(A_B,\ M_B)_A \mid M\otimes_B A_A \mid M_A$, and hence $B^*{}_{A^*}\cong \mathrm{Hom}(M_A,\ \mathrm{Hom}(A_B,\ M_B)_A)_{A^*} \mid \mathrm{Hom}(M_A,\ M_A)_{A^*}=A^*{}_{A^*}$. On the other hand, since $M\otimes_B A_A \mid M_A$, we have ${}_{A^*}\mathrm{Hom}(M\otimes_B A_A,\ M_A)_{B^*}\cong {}_{A^*}\mathrm{Hom}(\mathrm{Hom}(M_A,\ M\otimes_B A_A)_{A^*},\ \mathrm{Hom}(M_A,\ M_A)_{A^*})_{B^*}$ by [8 ; Lemma 3. 6 (4)], whence it follows ${}_{A^*}\mathrm{Hom}(B^*{}_{A^*},\ A^*{}_{A^*})_{B^*} \mid$ (resp. $\sim$)${}_{A^*}B^*{}_{B^*}$. Now, combining this with Lemma 1. 1, we see that B^*/A^* is a left QF-(resp. QF-) extension.

Corollary 1. 3. *Assume that* $A_A \mid M_A$ *and* $M_B \mid B_B$. *If* A/B *is a* QF-*(resp. left* QF-*) extension then* B^*/A^* *is a* QF- *(resp. left* QF-*) extension and* $A/\bar{B}$ *is a* QF-*extension, where* $\bar{B}=\mathrm{End}({}_{B^*}M)$.

Proof. By $M_B \mid B_B$ and $A_A \mid M_A$, there holds $M\otimes_B A_A \mid B\otimes_B A_A \cong A_A \mid M_A$. On the other hand, it is well known that ${}_{A^*}M \mid {}_{A^*}A^*$ and ${}_{B^*}B^* \mid {}_{B^*}M$. Thus, the corollary is immediate from Theorem 1. 2.

Corollary 1. 4. *Let* A/B *be a* QF-*(resp. left* QF-*) extension. If* B'

$=\mathrm{End}(A_B)$ *and* $\bar{B}=\mathrm{End}({}_{B'}A)$ *then* B'/A *is a QF-(resp. left QF-) extension and* $A/\bar{B}$ *is a QF-extension.*

A Frobenius extension A/B is said to be *symmetric* if ${}_{B,V_A(B)}A_A \cong {}_{B,V_A(B)}\mathrm{Hom}(A_B, B_B)_A$, where $V_A(B)$ is the centralizer of B in A and $\mathrm{Hom}(A_B, B_B)$ is regarded as a left $V_A(B)$-module by $(af)(x)=f(xa)(a\in V_A(B), x\in A, f\in \mathrm{Hom}(A_B, B_B))$. To be easily seen, a ring extension A/B is symmetric if and only if there exist h in $\mathrm{Hom}({}_BA_B, {}_BB_B)$, l_i, r_i in $A(i=1,\cdots,n)$ such that $h(xa)=h(ax)$ and $x=\sum_i r_ih(l_ix)=\sum_i h(xr_i)l_i$ for all x in A and a in $V_A(B)$ (see [11]). When this is the case, $(h\,;l_i, r_i)_i$ will be called a *symmetric system* for A/B.

Lemma 1 5. *Let* A/B *and* B/C *be ring extensions. If both* A/B *and* B/C *are symmetric and* $V_A(C)=V_A(B)\cdot V_B(C)$ *then* A/C *is symmetric.*

Proof. Let $(h\,;l_i, r_i)_i$ and $(h'\,;l'_j, r'_j)_j$ be respective symmetric systems for A/B and B/C. Then, it is easy to see that $(h'h\,;l'_jl_i, r_ir'_j)_{i,j}$ is a symmetric system for A/C.

Corollary 1. 6. *If* A/B *is symmetric then so is* $(A)_n/B$, *where* $(A)_n$ *denotes the* $n\times n$ *matrix ring over* A.

Proof. Let Tr be the canonical trace of $(A)_n$ to A, and e_{ij} the matrix in $(A)_n$ with 1 in the (i, j)-position and zeros elsewhere. Then, $(\mathrm{Tr}\,;e_{ij}, e_{ji})_{i,j}$ is a symmetric system for $(A)_n/A$, and hence $(A)_n/B$ is symmetric by Lemma 1. 5.

Proposition 1. 7. *Let* A *be an* R*-algebra which is faithful as* R*-module, and let* M_A *be a finitely generated, projective* A*-module which is faithful as* R*-module. If* A/R *is symmetric then so is* $\mathrm{End}(M_A)/R$.

Proof. Since $M_A \mid A_A$, $M_A\oplus M'_A=A^n_A$ for some n, and hence $\mathrm{End}(M_A)=e(A)_ne$ with some idempotent element e in $(A)_n$. Noting that $(A)_n/R$ is symmetric by Corollary 1. 6, there exists a symmetric system $(h\,;l_i, r_i)_i$ for $(A)_n/R$. Then, we can easily see that $(h\mid e(A)_ne\,;el_ie, er_ie)_i$ is a symmetric system for $e(A)_ne/R$, where $h\mid e(A)_ne$ denotes the restriction of h on $e(A)_ne$.

2. Throughout the present section, a ring extension A/B will be fixed.

Proposition 2. 1. *If* ${}_BA_A \mid {}_B\mathrm{Hom}(A_B, B_B)_A$ *and* X_B *is finitely gene-*

rated, projective, injective and faithful then so is $X \otimes_B A_A$.

Proof. Since X_B is finitely generated and projective, $X \otimes_B A_A | B \otimes_B A_A \cong A_A$, that is, $X \otimes_B A_A$ is finitely generated and projective. Next, ${}_B A_A | {}_B\mathrm{Hom}(A_B, B_B)_A$ and $X_B | B_B$ imply $X \otimes_B A_A | X \otimes_B \mathrm{Hom}(A_B, B_B)_A$ (canonically isomorphic to $\mathrm{Hom}(A_B, X_B)_A$). Since X_B is injective, $\mathrm{Hom}(A_B, X_B)_A$ is injective by [3 ; Chap. VI, § 1, Proposition 1. 4], and hence $X \otimes_B A_A$ is injective. Finally, assume that $(X \otimes_B A)a = 0 (a \in A)$. Then $X \otimes_B Aa = 0$ implies that $Xg(a) = 0$ for all g in $\mathrm{Hom}({}_B A, {}_B B)$. Accordingly, X_B being fatihful, $g(a) = 0$ for all g. As is well known, the dual ${}_B\mathrm{Hom}(A_B, B_B)$ of A_B is torsionless. Recalling here that ${}_B A | {}_B\mathrm{Hom}(A_B, B_B)$, it follows that ${}_B A$ is torsionless. Hence, we obtain $a = 0$.

Corollary 2. 2. *Let e be an idempotent element of B. If* ${}_B A_A | {}_B\mathrm{Hom}(A_B, B_B)_A$ *and* eB_B *is faithful and injective then so is* eA_A.

Proposition 2. 3. *Assume that* ${}_B\mathrm{Hom}(A_B, B_B)_A \subset {}_B A_A^n$ *and* $A_B | B_B$. *If* X_B *is finitely generated, projective, injective and faithful then so is* $\mathrm{Hom}(A_B, X_B)_A$.

Proof. Since ${}_B\mathrm{Hom}(A_B, B_B)_A \subset {}_B A_A^n$, $A_B | B_B$ and $X_B | B_B$, we have $\mathrm{Hom}(A_B, X_B)_A \cong X \otimes_B \mathrm{Hom}(A_B, B_B)_A \subset X \otimes_B A_A^n \cong (X \otimes_B A)_A^n | A_A^n$, where each isomorphism is canonical. Hence, the injectivity of X_B implies that $\mathrm{Hom}(A_B, X_B)_A$ is finitely generated, projective and injective. Finally, assume that $\mathrm{Hom}(A_B, X_B)a = 0 (a \in A)$. Then $\mathrm{Hom}(A_B, X_B)(a) = (\mathrm{Hom}(A_B, X_B)a)(1) = 0$. Since $A_B | B_B$ and X_B is faithful, it is easy to see that A_B can be embedded in some product ΠX_B of X_B's. It follows therefore $a = 0$.

Remark. If $A_A | \mathrm{Hom}(A_B, B_B)_A$ and X_B is faithful then $\mathrm{Hom}(A_B, X_B)_A$ is faithful.

The following is due to F. Kasch [6 ; Hilfssatz 4].

Lemma. *Assume that* ${}_B A$ *is projective and* A_B *is finitely generated and projective. If* Y_A *is injective (resp. projective) then so is* Y_B.

Now, combinining the above lemma with Propositions 2. 1 and 2. 3, one will readily obtain the following :

Theorem 2. 4. *Let* A/B *be a right or left QF-extension. If A is a right (resp. left) QF-3 ring then B is a right (resp. left) QF-3 ring, and conversely.*

Relating to the above discussion, we shall prove the following :

Proposition 2. 5. *Let R be a ring, M a right R-module which is completely faithful and torsionless, and $S=\mathrm{End}(M_R)$. If a right R-module X is finitely generated, projective, injective and faithful then so is* $\mathrm{Hom}(M_R, X_R)_S$. *In particular, S is a right QF-3 ring whenever R is a right QF-3 ring.*

Proof. As is well known, $R_R | M_R$ implies ${}_SM | {}_SS$. Accordingly, X_R being injective, it follows that $\mathrm{Hom}(M_R, X_R)_S$ is injective. On the other hand, $X_R \mid R_R \mid M_R$ yields $\mathrm{Hom}(M_R, X_R)_S \mid \mathrm{Hom}(M_R, M_R)_S = S_S$, that is, $\mathrm{Hom}(M_R, X_R)_S$ is finitely generated and projective. Finally, as M_R is torsionless and X_R is faithful, we see that $M_R \subsetneq \Pi X_R$. Hence, $S= \mathrm{Hom}(M_R, M_R)_S \subsetneq \mathrm{Hom}(M_R, \Pi X_R)_S \cong \Pi\,\mathrm{Hom}(M_R, X_R)_S$, whence it follows the faithfulness of $\mathrm{Hom}(M_R, X_R)_S$.

The next is due to T. Onodera [12]. For the sake of completeness, we shall give here the proof.

Proposition. *Assume that* $\mathrm{Hom}(A_B, B_B)_A$ *is torsionless. Let M_A be an A-module. If M_B is torsionless then so is M_A.*

Proof. Since $M_B \subsetneq \Pi B_B$ and the canonical homomorphism $M_A \ni m \mapsto (a \mapsto ma) \in \mathrm{Hom}(A_B, M_B)_A$ is one to one, we have $M_A \subsetneq \mathrm{Hom}(A_B, M_B)_A \subsetneq \mathrm{Hom}(A_B, \Pi B_B)_A \cong \Pi\ \mathrm{Hom}(A_B, B_B)_A$. It follows therefore M_A is torsionless, for $\mathrm{Hom}(A_B, B_B)_A$ torsionless.

Corollary 2. 6. *Under the same assumption as in the previous proposition, if B_B is a cogenerator in the category of right B-modules then A_A is a cogenerator in the category of right A-modules.*

Corollary 2. 7. *Let A/B be a left QF-extension. If B_B is a cogenerator then so is A_A. Conversely, if A_A is a cogenerator and $B_B | A_B$ then B_B is a cogenerator.*

Proof. Since $\mathrm{Hom}(A_B, B_B)_A \mid A_A$, $\mathrm{Hom}(A_B, B_B)_A$ is torsionless. Accordingly, the first assertion is contained in Corollary 2. 6. Next, we shall prove the second assertion. Let $B'=\mathrm{End}(A_B)$. Then, B'/A is a left QF-extension by Corollary 1. 4. Since A_A is a cogenerator, $B'_{B'}$ is then a cogenerator by the first assertion. On the other hand, noting that $A_B \sim B_B$, we see that the category of right B-modules is isomorphic to the category of right B'-modules, from which it follows that B_B is a cogenerator.

Proposion 2. 8. *Assume that* ${}_B\mathrm{Hom}(A_B, B_B)_A \mid {}_BA_A$ *and A_B is finitely generated. If B is a left S-ring then so is A.*

Proof. One may remark that $\mathrm{Hom}(N_B, B_B) \neq 0$ for every finitely generated non-zero B-module N_B. Now, let M_A be an arbitrary simple A-module. Then A_B is finitely generated, and so $\mathrm{Hom}(M_B, B_B) \neq 0$. On the other hand, there holds ${}_B\mathrm{Hom}(M_B, B_B) \cong {}_B\mathrm{Hom}(M \otimes_A A_B, B_B) \cong {}_B\mathrm{Hom}(M_A, \mathrm{Hom}(A_B, B_B)_A) \mid {}_B\mathrm{Hom}(M_A, A_A)$, where each isomorphism is canonical. Hence, $\mathrm{Hom}(M_A, A_A) \neq 0$, that is, A is a left S-ring.

Corollary 2. 9. *Let A/B be a left QF-extension. If B is a left S-ring then* so *is A. Conversely, if B is a left S-ring and $B_B \mid A_B$ then A is a left S-ring.*

Proof. The first assertion is a direct consequence of Proposition 2. 8 and the proof of the second one is quite similar to that of Corollary 2. 7.

In what follows, $\widehat{M}_R$ will represent the injective hull of M_R(see [4]).

Lemma 2. 10. (a) *Let $\mathrm{Hom}(A_B, B_B)_A$ be faithful and torsionless. If $\widehat{B}_B$ is torsionless then so is $\widehat{A}_A$.*

(b) *Let $A_A \subset \mathrm{Hom}(A_B, B_B)^m_A \subset A^n_A$ for some m, n. If $\widehat{B}_B$ is finitely generated and projective, then so is $\widehat{A}_A$.*

Proof. One may remark that $\mathrm{Hom}(A_B, B_B)_A \subset \mathrm{Hom}(A_B, \widehat{B}_B)_A$ and $\mathrm{Hom}(A_B, \widehat{B}_B)_A$ is injective.

(a) Since $\widehat{B}_B$ and $\mathrm{Hom}(A_B, B_B)_A$ are torsionless, it follows $\mathrm{Hom}(A_B, \widehat{B}_B)_A \subset \mathrm{Hom}(A_B, \Pi\, B_B)_A \cong \Pi\, \mathrm{Hom}(A_B, B_B)_A \subset \Pi\, A_A$. On the other hand, $\mathrm{Hom}(A_B, B_B)_A$ being faithful, we have $A_A \subset \Pi\, \mathrm{Hom}(A_B, B_B)_A \subset \Pi\, \mathrm{Hom}(B_B, \widehat{B}_B)_A$. Hence, $\widehat{A}_A \subset \Pi\, \mathrm{Hom}(A_B, \widehat{B}_B)_A$, which means that $\widehat{A}_A$ is torsionless.

(b) Since $\widehat{B}_B$ is finitely generated and projective, $\mathrm{Hom}(A_B, \widehat{B}_B)_A \mid \mathrm{Hom}(A_B, B_B)_A \subset A^n_A$. Hence, the injective module $\mathrm{Hom}(A_B, \widehat{B}_B)_A$ is finitely generated and projective. On the other hand, $A_A \subset \mathrm{Hom}(A_B, B_B)^m_A \subset \mathrm{Hom}(A_B, \widehat{B}_B)^m_A$, and so $\widehat{A}_A \mid \mathrm{Hom}(A_B, \widehat{B}_B)_A$. It follows therefore $\widehat{A}_A$ is finitely generated and projective.

Remark. If $\widehat{R}_R$ is projective then $\widehat{R}_R$ is finitely generated (see [5; Proposition 2. 4]).

The next will be easily seen;

Lemma 2. 11. *Let M_A be an A-module.*

(a) *If $A_B \subset \Pi B_B$ and $M_A \subset \Pi A_A$ then $M_B \subset \Pi B_B$.*

(b) *If A_B and M_A are flat then M_B is flat.*

Proposition 2. 12. *Let A/B be a QF-extension. Then, $\widehat{A}_A$ is flat (resp. torsionless, finitely generated and projective) if and only if $\widehat{B}_B$ is flat (resp. torsionless, finitely generated and projective).*

Proof. Since A/B is a QF-extension, ${}_BA_A \sim {}_B\mathrm{Hom}(A_B, B_B)_A$, $A_B | B_B$ and ${}_BA \,|\, {}_BB$. Then, by Kasch's lemma, the injectivity of $\widehat{A}_A$ implies that of $\widehat{A}_B$, and so $\widehat{B}_B < \oplus \widehat{A}_B$. Thus, the only if part is an immediate consequence of Lemma 2. 11. Let us assume now that $\widehat{B}_B$ is flat. Then, $\widehat{B} \otimes_B A_A$ is flat and $A_A \,|\, \mathrm{Hom}(A_B, B_B)_A \subset \mathrm{Hom}(A_B, \widehat{B}_B)_A \cong \widehat{B} \otimes_B \mathrm{Hom}(A_B, B_B) \,|\, \widehat{B} \otimes_B A_A$. It follows therefore $\widehat{A}_A$ is flat. The remaining is immediate by Lemma 2. 10.

Proposition 2. 13. *Assume $A_A < \oplus \Pi \mathrm{Hom}(A_B, B_B)_A \subset \Pi A_A$. If B is right PF then so is A.*

Proof. By T. Kato [7; Theorem 7], a ring R is right PF if and only if R_R is injective and cogenerator. If B is right PF then B_B, and hence $\mathrm{Hom}(A_B, B_B)_A$ is injective. Recalling here that $A_A < \oplus \Pi \mathrm{Hom}(A_B, B_B)_A$, we see that A_A is injective. On the other hand, A_A is a cogenerator by Corollary 2. 6.

Corollary 2. 14. *Let A/B be a QF-extension. If B is right PF then so is A. Conversely, if B is right PF and $B_B | A_B$ then A is right PF* (cf. Sugano [13]).

Proof. The first assertion is a direct consequence of Proposition 2. 13, and the proof of the second one is quite similar to that of Corollary 2. 7.

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DEPARTMENT OF MATHEMATICS,
HOKKAIDO UNIVERSITY

(*Received February 15, 1971*)