

GROUP RINGS WITH SOLVABLE UNIT GROUPS

Dedicated to Professor TAKESHI INAGAKI on the occasion
of his sixtieth birthday

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Throughout, R will represent an artinian simple ring with 1, C the center of R , and G a finite group of order g .

Recently, in their paper [1], P. B. Bhattacharya and S. K. Jain proved the following : Let S be a right artinian ring with 1, and $J(S)$ the (Jacobson) radical of S . Then, the unit group $U(S)$ of S is solvable if and only if $U(S/J(S))$ is solvable¹⁾, namely, $S/J(S)$ is a finite direct sum of fields, $(\text{GF}(2))_2$ and $(\text{GF}(3))_3$. This result will be used freely in the sequel. The main theme of our discussion will concern the solvability of the unit group of the group ring RG .

For semi-simple group rings, we shall prove the following :

Theorem 1. *Assume that g is not divisible by the characteristic of C . Then, $U(RG)$ is solvable if and only if there holds one of the following :*

- (i) $R=C$ and G is abelian (or equivalently, $U(RG)$ is nilpotent²⁾).
- (ii) $R=(\text{GF}(2))_2$ and $G=1$.
- (iii) $R=(\text{GF}(3))_2$ and G is an abelian group of exponent 2.
- (iv) $R=\text{GF}(3)$ and $G=G_1 \times G_2$, where G_1 is an abelian group of exponent 2 and G_2 is a non-abelian indecomposable 2-group such that $U(RG_2)$ is solvable.

Proof. Assume first $U(RG)$ is solvable. Since $U(R)$ is solvable, $R=C$ or $(\text{GF}(2))_2$ or $(\text{GF}(3))_3$. Now, we shall distinguish between three cases :

1) More generally, there holds the following : Let S be a ring with 1, and N a nilpotent ideal of S . If $U(S/N)$ is solvable then $U(S)$ is solvable (and conversely). To see this, we consider the group homomorphism $\alpha: U(S) \rightarrow U(S/N)$ induced by the natural ring homomorphism $S \rightarrow S/N$. Let $(\text{Ker } \alpha)^{[n]}$ be the n -th commutator subgroup of $\text{Ker } \alpha$. Noting that $a^{-1}b^{-1}ab-1 = a^{-1}b^{-1}((a-1)(b-1)-(b-1)(a-1))$ for $a, b \in U(S)$, we can easily see $\{u-1 \mid u \in (\text{Ker } \alpha)^{[n]}\} \subset N^{2^n}$, which implies the solvability of $\text{Ker } \alpha$.

2) Cf. [3].

Case 1. $R = C \neq \text{GF}(3)$: It suffices then to consider the case $R = \text{GF}(2)$: $RG = F_1 \oplus \cdots \oplus F_k \oplus (\text{GF}(2))_2^{(g)}$, where F_i are fields over $\text{GF}(2)$ and $(\text{GF}(2))_2^{(g)}$ means the direct sum of l copies of $(\text{GF}(2))_2$. Since $\text{GL}(2, 2)$ is isomorphic to the symmetric group S_3 and the alternative group A_3 is abelian, we obtain a normal chain $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_{n-1} \triangleright G_n$ such that $(G_{i-1} : G_i) \leq 2$ and G_n is abelian. Noting that g is odd, we readily see that G is equal to the abelian group G_n .

Case 2. $R = (\text{GF}(2))_2$ or $(\text{GF}(3))_2$: Evidently, $RG = F_1 \oplus \cdots \oplus F_k \oplus R^{(g)}$. Since $U(CG)$ is solvable and $RG = (CG)_2$, one will easily see that $CG = C^{(g)}$. If $R = (\text{GF}(2))_2$ then $U(CG) = 1$, and hence $G = 1$. On the other hand, if $R = (\text{GF}(3))_2$ then $U(CG)$ is an abelian group of exponent 2, and hence so is G .

Case 3. $R = \text{GF}(3)$: Assume G is non-abelian. Then, $G = G_1 \times G_2$, where G_2 is non-abelian and indecomposable. Since both $U(RG_1)$ and $U(RG_2)$ are solvable, $RG_1 = F'_1 \oplus \cdots \oplus F'_{k'} \oplus (R)_2^{(l')}$ and $RG_2 = F''_1 \oplus \cdots \oplus F''_{k''} \oplus (R)_2^{(l'')}$ where F'_i and F''_j are fields over R and $l'' > 0$. Accordingly, RG is the direct sum of $\bigoplus_{i,j} (F'_i \otimes_R F''_j)$, $\bigoplus_i (F'_i)^{(l')}$, $\bigoplus_j (F''_j)^{(l'')}$ and $(R)_2^{(l'l'')}$. It follows then $l' = 0$ and $F'_i = R$. Hence, G_1 is an abelian group of exponent 2. Now, let G_2^* be the residue class group of G_2 modulo its center. Since G_2^* can be regarded as a subgroup of the direct product of l'' copies of $\text{PGL}(2, 3)$ and $\text{PGL}(2, 3) (\cong S_4)$ is of order $2^3 \cdot 3$, G_2^* is a non-trivial 2-group, and hence G_2 is nilpotent. Recalling here that G_2 is indecomposable, we readily see that G_2 is a 2-group.

Concerning the converse, it suffices to prove that if (iii) or (iv) holds then $U(RG)$ is solvable. In fact, if there holds (iii) then $CG = C^{(g)}$, whence we readily see that $RG = (C)_2^{(g)}$ and $U(RG)$ is solvable. Next, if there holds (iv) then $RG_i = R^{(g_i)}$ (g_i the order of G_i) and $U(RG) = U(RG_1 \otimes_R RG_2)$ is solvable.

Remark. Let $R = \text{GF}(3)$. If $g = 8$ then $U(RG)$ is solvable. In fact, if G is non-abelian then the commutator subgroup G' of G coincides with the center of G and G/G' is a 2-elementary abelian group of order 4. Accordingly, G possesses four 1 dimensional representations in R , and it is easy to see that $RG = R^{(4)} \oplus (R)_2$. However, in case $g = 16$, there exists an indecomposable G such that $U(GR)$ is not solvable: Let G be the group $\langle a, b \rangle$ with defining relations $a^8 = 1$, $b^2 = 1$, $b^{-1}ab = a^7$. To be easily seen, G is indecomposable and possesses the irreducible representation T in R

defined by $T(a) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ and $T(b) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. The commutator of $\{T(a), T(b)\}$ in $(R)_4$ is seen to be $R(\zeta)$, where $\zeta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is a primitive

fourth root of 1. Hence, RG contains $(R(\zeta))_2$ as a simple component, and so $U(RG)$ is not solvable.

Theorem 2. *Assume that C is of prime characteristic p and G contains a normal Sylow p -subgroup P . Then, $U(RG)$ is solvable if and only if there holds one of the following :*

- (i) $R=C$ and G is a semi-direct product of P and an abelian group.
- (ii) $R=(GF(2))_2$ and $G=P$.
- (iii) $R=(GF(3))_2$ and G is a semi-direct product of P and an abelian group of exponent 2.
- (iv) $R=GF(3)$ and G is a semi-direct product of P and $G_1 \times G_2$, where G_1 is an abelian group of exponent 2 and G_2 is a non-abelian indecomposable 2-group such that $U(RG_2)$ is solvable.

Proof. We consider the ring epimorphism $\beta: RG \rightarrow RG^*$ defined by $\sum_{\sigma \in G} a_\sigma \sigma \mapsto \sum_{\sigma \in G} a_\sigma \sigma^*$, where $G^* = G/P$ and σ^* is the residue class of σ modulo P . Then, by [2; Theorem 2], $\text{Ker } \beta$ coincides with $J(RG)$. Hence, $U(RG)$ is solvable if and only if so is $U(RG^*)$. Now, our assertion will be immediate by Theorem 1 and Schur-Zassenhaus theorem.

The following corollaries are only selections from Theorems 1 and 2.

Corollary 1. *Let $C \neq GF(3)$.*

- (a) *Assume that $G \neq 1$ and g is not divisible by the characteristic of C . If $U(RG)$ is solvable then it is nilpotent.*
- (b) *Assume that C is of prime characteristic p and G contains a proper normal Sylow p -subgroup P . If $U(RG)$ is solvable then $U(R \cdot G/P)$ is nilpotent, and conversely.*

Corollary 2. *Let $C=GF(3)$.*

- (a) *Assume that G is not a 2-group and g is not divisible by 3. If $U(RG)$ is solvable then it is nilpotent.*

(b) Assume that G contains a normal Sylow 3-subgroup P such that G/P is not a 2-group. If $U(RG)$ is solvable then $U(R \cdot G/P)$ is nilpotent, and conversely.

Corollary 3. Let R be of prime characteristic p , and G a p -group. Then, $U(RG)$ is solvable if and only if $R=C$ or $(\text{GF}(2))_2$ or $(\text{GF}(3))_2$.

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