ON THE RADICAL OF A GROUP RING

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Let G be a finite group such that for the set $\{P=P_1, P_2, \cdots, P_k\}$ of its p-Sylow subgroups, $N_G(P)=P$ and $P_i \cap P_j=1$ for $i \neq j$. Then, by the operation $P_i^x=x^{-1}P_i x$ $(x \in G)$, G is a Frobenius group as a permutation group on $\{P_1, P_2, \cdots, P_k\}$ and a semi-direct product of P and its Frobenius kernel N (cf. for instance [4, Th. 17. 1]). Further, A will represent a semi-primary ring with 1 such that the center K of $A/J(A)$ ($J(A)$ the Jacobson radical of A) contains the prime field of characteristic p. We shall notice that $e=|N|^{-1}\sum_{\eta \in N}\eta$ is a central idempotent of the group ring AG and $J(AP)=\{\sum_{\sigma \in G} a_\sigma \sigma \mid \sum_{\sigma \in G} a_\sigma \in J(A)\}$ ([2, Cor. 1]). The purpose of this paper is to prove the following theorem.

Theorem. $J(AG)=J(AP)e+J(A)G$.

Proof. It is easy to verify that $J(AP)e$ is contained in $J(AG)$. Moreover, by [3, Th. 46. 2], $J(A)G$ is contained in $J(AG)$, and hence $J(AP)e+J(A)G \subseteq J(AG)$. Now, we shall prove the converse inclusion.

Step 1: Let A be a division ring. Since $J(AG)=J(A \otimes_K KG)=A \otimes_K J(KG)$ ([1, Th. 5. 6. 1]), it suffices to prove the case $A=K$. Let L be an algebraically closed field containing K. Then, [5, Th. 2] proves $[J(LG):L]=|P|-1$. Combining this with $[L \cdot J(KP)e: L]=[J(KP):K]=|P|-1$, we readily obtain $J(LG)=L \cdot J(KG)=L \cdot J(KG)e$, and hence $J(KG)=J(KP)e$.

Step 2: Let A be a simple ring: $A=(D)_n$ with a division ring D. Evidently, $\sum_{\sigma \in G}(d_{ij}^{(\sigma)})\sigma \mapsto (\sum_{\sigma \in G} d_{ij}^{(\sigma)}\sigma)$ defines a ring isomorphism h of AG onto $(DG)_n$. Then, by Step 1, $h(J(AG))=(J(DG))_n=(J(DP)e)_n=(J(DPe))_n=J((DPe)_n)=h(J((D)_n Pe))$, and hence it follows $J(AG)=J((D)_n Pe)=J((D)_n P)e$.

Step 3: Let $\bar{A}=A/J(A)=\bigoplus_{i=1}^m A_i$, where A_i is an artinian simple ring of characteristic p. Then, by Step 2, we obtain $J(\bar{A}G)=\bigoplus_i J(A_i G)=\bigoplus_i J(A_i P)\bar{e}, = J(\bar{A}P)\bar{e}$ where $\bar{e}=|N|^{-1}\sum_{\eta \in N}\eta$ considered in $\bar{A}G$. Evidently, $\sum_{\sigma \in G} a_\sigma \sigma \mapsto \sum_{\sigma \in G} \bar{a}_\sigma \sigma$ defines a ring epimorphism t of AG onto $\bar{A}G$, where $\bar{a}_\sigma$ means the residue class of a_σ modulo $J(\bar{A})$. Then, $t(J(AG)) \subseteq J(\bar{A}G)=J(\bar{A}P)\bar{e}$ and $t(e)=\bar{e}$. Hence, $J(AG) \subseteq J(AP)e+J(A)G$.

REFERENCES

[1] N. JACOBSON: Structure of Rings, Providence, 1956.
[2] K. MOTOSE: On group rings over semi-primary rings, Math. J. Okayama Univ. **14** (1969), 23—26.
[3] T. NAKAYAMA and G. AZUMAYA: Algebra II (Theory of Rings), Tokyo, 1954 (in Japanease).
[4] D. S. PASSMAN: Permutation Groups, Benjamin, 1968.
[5] D. A. R. WALLACE: Note on the radical of a group algebra, Proc. Cambr. Phil. Soc. **54** (1958), 128—130.

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