ON THE SUPPORT FUNCTIONS AND SPHERICAL SUBMANIFOLDS

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Let M be an n-dimensional submanifold of a euclidean space E^m of dimension m $(m>n)$. If $\boldsymbol{e}$ is a unit normal vector field of M and $\boldsymbol{X}$ is the position vector field, then the *support function with respect to* $\boldsymbol{e}$ is defined to be the scalar product $\boldsymbol{X}\cdot\boldsymbol{e}$ of $\boldsymbol{X}$ and $\boldsymbol{e}$. In § **2**, we study support functions and get some characterizations of spherical submanifolds. In §**3**, we derive two integral formulas for submanifolds which generalize the well-known formula of Minkowski for hypersurfaces. In the last section, we find two theorems analogous to the well-known characterization of a sphere in E^3 by Scherrer.

1. Preliminaries

Let M be an n-dimensional submanifold of a euclidean space E^m of dimension m. Let $F(M)$ and $F(E^m)$ be the bundles of orthonormal frames of M and E^m respectively. Let B be the set of elements $b=(p, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n, \cdots, \boldsymbol{e}_m)\in F(E^m)$ such that $(p, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n)\in F(M)$.

Throughout this paper, we shall argee the indices of the following ranges unless otherwise stated :

$$1\leqq i, j, \cdots\leqq n\,;\ 1\leqq A, B, \cdots\leqq m\,;\ n+1\leqq r, s, \cdots\leqq m.$$

The structure equations of E^m are given by

$$\begin{aligned} &dx=\sum \omega'_A\boldsymbol{e}_A, && d\boldsymbol{e}_A=\sum \omega'_{AB}\boldsymbol{e}_B, \\ &d\omega'_A=\sum \omega'_B\wedge\omega'_{BA}, && d\omega'_{AB}=\sum \omega'_{AC}\wedge\omega'_{CB}, \\ &\omega'_{AB}+\omega'_{BA}=0, \end{aligned} \tag{1}$$

where ω'_A, ω'_{AB} are differential 1-forms on $F(E^m)$. Let ω_A, ω_{AB} be the induced 1-forms on B from ω'_A, ω'_{AB} by the inclusion mapping $B\to F(E^m)$. Then we have $\omega_r=0$. Hence, by (1), we obtain

$$\sum \omega_i\wedge\omega_{ir}=0.$$

By a lemma of Cartan, we may write

$$\omega_{ir}=\sum A_{rij}\omega_j, \qquad A_{rij}=A_{rji}. \tag{2}$$

The *mean curvature vector* $\boldsymbol{H}$ is defined by

(3) $$\boldsymbol{H}=\left(\frac{1}{n}\right)\sum A_{rii}\boldsymbol{e}_r.$$

For each unit normal vector $\boldsymbol{e}=\sum\cos\theta_r\,\boldsymbol{e}_r$, the second fundamental form $A_{(e)}=(A_{ij}(\boldsymbol{e}))$ at $\boldsymbol{e}$ is the linear transformation given by

$$A_{(e)}(\boldsymbol{e}_i)=\sum\cos\theta_r A_{rij}\boldsymbol{e}_j,\qquad i=1,\cdots,n.$$

The principal curvatures; $k_1(\boldsymbol{e}),\cdots,k_n(\boldsymbol{e})$ at $\boldsymbol{e}$ are defined as the eigenvalues of the second fundamental form $A_{(e)}$ at $\boldsymbol{e}$. The *i-th mean curvature at* $\boldsymbol{e}$, $K_i(p,\boldsymbol{e})$, is given by the i-th elementary symmetric function divided by $\binom{n}{i}=n!/i!(n-i)!$, i.e.,

(4) $$\binom{n}{i}K_i(p,\boldsymbol{e})=\sum k_1(\boldsymbol{e})\cdots k_i(\boldsymbol{e}).$$

Definition 1. Let $\boldsymbol{e}$ be a unit normal vector field of M in E^m. If the n-th mean curvature $K_n(p,\boldsymbol{e})\neq 0$ for all $p\in M$ except on a measure zero subset of M, then the normal vector field $\boldsymbol{e}$ is called a *non-degenerate normal vector field of M.*

Definition 2. If M is contained in a hypersphere of E^m centered at the origin of E^m, then M is called a *spherical submanifold* in E^m.

Definition 3. A unit normal vector field $\boldsymbol{e}$ of M in E^m is said to be *parallel in the normal bundle* if $d\boldsymbol{e}$ is tangent to M everywhere.

2. Submanifolds with constant support function

Proposition 1. *Let M be an n-dimensional submanifold of* E^m, $\boldsymbol{e}$ *be a unit normal vector field of M in* E^m *and parallel in the normal bundle. If the support function* $\boldsymbol{X}\cdot\boldsymbol{e}$ *is equal to a constant, then M is the union of some spherical submanifolds with a subset W of M such that the n-th mean curvature at* $\boldsymbol{e}$ *vanishes identically on W.*

Proof. Let $\boldsymbol{e}_1,\cdots,\boldsymbol{e}_n$ be the principal directions of $\boldsymbol{e}$, then, by the assumption of the parallelism of $\boldsymbol{e}$ in the normal bundle, we have

(5) $$d\boldsymbol{e}=-\sum k_i(\boldsymbol{e})\omega_i\boldsymbol{e}_i.$$

Thus we obtain

(6) $$0=d(X\cdot e)=X\cdot de=-\sum k_i(e)\,(X\cdot e_i)\omega_i.$$

Therefore, on the set $U=\{p\in M:\ K_n(p, e)\neq 0\}$, we have

$$X\cdot e_1=X\cdot e_2=\cdots=X\cdot e_n=0,$$

i. e., X is normal to M on U. Hence we get $d(X\cdot X)=0$. This shows that each component of U is a spherical submanifold of E^m. This completes the proof of the proposition.

From Proposition 1, we have the following theorem:

Theorem 2. *Let M be an n-dimensional submanifold of E^m. If there exists a non-degenerate normal vactor field e such that e is parallel in the normal bundle and the support function $X\cdot e$ with respect to e is a constant, then M is a spherical submanifold of E^m.*

Proof. By the assumption that the unit normal vector field e is non-degenerate, we see that the subset W in Proposition 1 is a subset of measure zero in M. Hence, by Proposition 1, we get the theorem.

For a submanifold M in E^m, the position vector field X can be decomposed into two parts; $X=X_t+X_n$, where X_t is tangent to M and X_n is normal to M. If we denote the unit normal vector field in the direction of X_n by $\bar{e}$, i. e.,

(7) $$X_n=f\bar{e},$$

then f is the support function with respect to $\bar{e}$. We call this support function f the *canonical support function* of M in E^m.

Theorem 3. *Let M be a submanifold of E^m. If the canonical support function f is a nonzero constant and the last mean curvature $K_n(p, \bar{e})$ $(X_n=f\bar{e})$ with respect to $\bar{e}$ is not identically zero, then M is a spherical submanifold of E^m.*

Proof. Since the canonical support function f is a nonzero constant, we see that the normal component X_n of X is nowhere zero. Hence, we can choose $\bar{e}$ as a globally defined unit normal vector field on M. Let $e_1, \cdots, e_n$ be in the principal directions of $\bar{e}$, and $k_1, \cdots, k_n$ be the principal curvatures at $\bar{e}$. Then, if we choose $\bar{e}$ as the first unit normal vector field e_{n+1}, then we obtain

(8) $$d\bar{e}=-\sum k_i\omega_i e_i+\sum \omega_{n+1,\,r}\, e_r.$$

and

(9) $$e_r\cdot X=e_r\cdot X_t+e_r\cdot X_n=fe_r\cdot e_{n+1}=0,\quad \text{for } r=n+2,\cdots,m.$$

From (8) and (9) we obtain

(10) $$0=df=d(X\cdot\bar{e})=X\cdot d\bar{e}=-\sum k_i(X\cdot e_i)\omega_i.$$

Let U be the open subset of M; $U=\{p\in M:\ K_n(p,\bar{e})\neq 0\}$. Then U is not empty by the assumption. Moreover, by (10), we obtain $X\cdot e_1=\cdots=X\cdot e_n=0$. This implies $d(X\cdot X)=0$, on U. Hence, if let U^* denote a component of U, then U^* is a spherical submanifold of E^m, and the position vector field X on U^* satisfies $X=f\bar{e}$. The last statement implies that $K_n(p,\bar{e})$ is a constant $\left(=\frac{1}{(-f)^n}\right)$. Therefore, we see that U^* is a closed subset of M. Consequently, $M=U^*=U$, and M is a spherical submanifold of E^m. This completes the proof of the theorem.

Remark 1. The assumption of the non-vanishing of $K_n(p,\bar{e})$ is essential. Because an n-dimensional linear subspace of E^m has constant canonical support function f, and $f\neq 0$ if this subspace does not pass through the origin of E^m.

Corollary 1. *Let M be a closed hypersurface of E^{n+1}, and e be a unit normal vector field of M in E^{n+1}. If the support function $X\cdot e$ is a constant, then M is a hypersphere of E^{n+1}.*

Proof. Since M is a closed hypersurface of E^{n+1}, the support function $X\cdot e$ is just the canonical support functon and the n-th mean curvature $K_n(p,e)$ is not zero somewhere (see, for instance [1]). Hence, by Theorem 3, we get the corollary.

Remark 2. If M is a spherical submanifold of E^m, then the unit normal vector field $e=X/|X|$ is a non-degenerate normal vector field, parallel in the normal bundle, the support function $X\cdot e$ with respect to e is constant and the last mean curvature at e is a nonzero constant.

Remark 3. The unit normal vector field e satisfying the assumptions of Theorem 2 is not unique, in general. For example; let M^2 be a standard flat torus in E^4 given by

(11) $$(a\cos u,\ a\sin u,\ b\cos v,\ b\sin v),\quad a>0,\ b>0.$$

Then for every t such that $t \not\equiv 0 \left(\bmod \frac{\pi}{4}\right)$, the unit normal vector field

$$e_t = \frac{\cos t}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v) + \frac{\sin t}{\sqrt{2}}(\cos u, \sin u, -\cos v, -\sin v)$$

is a non-degenerate normal vector field, parallel in the normal bundle and the support function $X\cdot e_t$ with respect to e_t is constant.

3. Integral formulas for submanifolds and their applications

Let e be a unit normal vector field and u, v be two vector fields over M. Put

$$B(e_i, e_j) = \sum A_{ij}(e_r)e_r, \tag{12}$$

and

$$F(u, v) = \left(\frac{1}{n}\right)\sum_{i,j}(B(e_i, e_j)\cdot u)(B(e_i, e_j)\cdot v). \tag{13}$$

Then $B(e_i, e_j)$ and $F(u, v)$ are well-defined.

Suppose f is a smooth function on M. By **grad** f or ∇f, we mean $\nabla f = \sum f_i e_i$, where f_i are given by $df = \sum f_i \omega_i$.

Theorem 4. *Let M be an n-dimensional oriented closed submanifold of E^m, and e be a unit normal vector field parallel in the normal bundle. Then we have*

$$\int_M K_1(p, e) + F(e, X) + X\cdot \nabla K_1(p, e))dV = 0. \tag{14}$$

Proof. Since e is parallel in the normal bundle, we obtain

$$\begin{aligned} &d(X\cdot e) = -\sum A_{ij}(e)(X\cdot e_i)\omega_j \text{ and} \\ &A_{ij;k}(p, e) = A_{ik;j}(p, e). \end{aligned} \tag{15}$$

Apply the Hodge star operator $*$ on (15) we obtain

$$*d(X\cdot e) = \sum(-1)^j A_{ij}(e)(X\cdot e_i)\omega_1\wedge\cdots\wedge\hat{\omega}_j\wedge\cdots\wedge\omega_n. \tag{16}$$

Hence, the Laplacian $\triangle(X\cdot e)$ is given by

$$\begin{aligned} \triangle(X\cdot e)dV = d*d(X\cdot e) &= -(nK_1(p, e) + \sum A_{ij}(e)A_{ij}(e_r)(X\cdot e_r) \\ &\quad + \sum A_{ij;j}(p, e)(X\cdot e_i))dV \\ &= -(nK_1(p, e) + nF(e, X) + \sum A_{jj;i}(p, e)(X\cdot e_i))dV. \end{aligned}$$

From this we obtain

$$(17)\qquad \triangle(\boldsymbol{X}\cdot\boldsymbol{e}) = -n(K_1(p, \boldsymbol{e}) + F(\boldsymbol{e}, \boldsymbol{X}) + \boldsymbol{X}\cdot \nabla K_1(p, \boldsymbol{e})).$$

Integrating (17) over M and applying Green's theorem to the left hand side, we obtain (14). This completes the proof of the theorem.

By (17) and Hopf's lemma we obtain the following two corollaries.

Corollary 1. *Let M be a closed submanifold of E^m, and $\boldsymbol{e}$ a unit normal vector field parallel in the normal bundle. If $\boldsymbol{e}$ is non-degenerate and either $K_1(p, \boldsymbol{e}) + F(\boldsymbol{e}, \boldsymbol{X}) + \boldsymbol{X}\cdot\nabla K_1(p, \boldsymbol{e}) \geqq 0$ or $K_1(p, \boldsymbol{e}) + F(\boldsymbol{e}, \boldsymbol{X}) + \boldsymbol{X}\cdot\nabla K_1(p, \boldsymbol{e}) \leqq 0$ everywhere, then M is a spherical submanifold.*

Corollary 2. *Let M be a closed submanifold of E^m, and $\boldsymbol{e}$ be a unit normal vector field parallel in the normal bundle. Then the mean curvature vector $\boldsymbol{H}$ is perpendicular to $\boldsymbol{e}$ if and only if $K_1(p, \boldsymbol{e})$ is a constant and $\int_M F(\boldsymbol{e}, \boldsymbol{X})\, dV = 0$.*

Remark 4. If $\boldsymbol{e}$ is in the direction given by (7), then the condition of the parallelism of $\boldsymbol{e}$ in the normal bundle in Theorem 4, and Corollaries 1 and 2 can be omitted.

Theorem 5. *Under the same hypothesis of Theorem 4, we have*

$$(18)\qquad (n-1)\int_M K_1(p, \boldsymbol{e})dV + n\int_M (\boldsymbol{X}\cdot\boldsymbol{H})K_1(p, \boldsymbol{e})dV = \int_M F(\boldsymbol{e}, \boldsymbol{X})dV.$$

Proof. Let

$$(19)\qquad \theta = \sum(-1)^{i-1}(\boldsymbol{X}\cdot\boldsymbol{e}_i)\omega_1\wedge\cdots\wedge\hat{\omega}_i\wedge\cdots\wedge\omega_n,$$

where $\hat{}$ denotes the omitted term. Then, we get

$$(20)\qquad d\theta = n(1 + (\boldsymbol{X}\cdot\boldsymbol{H}))dV.$$

Hence, by the fact $dK_1(p, \boldsymbol{e})\wedge\theta = (\boldsymbol{X}\cdot\nabla K_1(p, \boldsymbol{e}))dV$, we obtain

$$(21)\qquad d(K_1(p, \boldsymbol{e})\theta) = (\boldsymbol{X}\cdot\nabla K_1(p, \boldsymbol{e}))dV + nK_1(p, \boldsymbol{e})(1 + (\boldsymbol{X}\cdot\boldsymbol{H}))dV.$$

Integrating (21) over M and applying Green's theorem, we obtain

$$(22)\qquad \int_M \{\boldsymbol{X}\cdot\nabla K_1(p, \boldsymbol{e}) + nK_1(p, \boldsymbol{e})[1 + (\boldsymbol{X}\cdot\boldsymbol{H})]\}dV = 0.$$

Combining (14) and (22) we obtain (16).

Corollary 1 (Minkowski's formula). *Let M be an oriented closed hypersurface of E^{n+1}, and e be the unit outer normal vector field. Then we have*

$$\int_M K_1(p, e)dV + \int_M (X\cdot e)K_2(p, e)dV = 0. \tag{23}$$

Proof. Since M is a hypersurface of E^{n+1}, every unit normal vector field is parallel in the normal bundle and

$$(X\cdot H)K_1 = (X\cdot e)K_1^2, \quad F(e, X) = (X\cdot e)(nK_1^2-(n-1)K_2), \tag{24}$$

where $K_i = K_i(p, e)$, $i=1, 2$. Hence, by (18) and (24), we get (23).

If M is a minimal submanifold of E^m, i. e., the mean curvature vector $H=0$ identically, then for every fixed vector c in E^m, we have $c\cdot H=0$ identically. On the other hand, when M is closed we have

Proposition 6. *Let M be a closed submanifold of E^m, and c be a non-zero vector in E^m. If we have either $c\cdot H\geqq 0$ or $c\cdot H\leqq 0$ everywhere on M, then M is contained in an $(m-1)$-dimensional linear subspace of E^m whose normal in E^m is parallel to c.*

Proof. By a direct computation for the Laplacian of $c\cdot X$, we have

$$\triangle(X\cdot c) = nc\cdot H. \tag{25}$$

Hence, if we have either $c\cdot H\geqq 0$ or $c\cdot H\leqq 0$ everywhere on M, then we obtain $c\cdot X=$constant. This implies that M is contained in an $(m-1)$-dimensional linear subspace of E^m whose normal in E^m is parallel to c.

4. Two theorems of Scherrer's type

In [3], Scherrer proved that a closed surface M^2 in E^3 is a sphere when and only when for every closed smooth curve on M^2 the integral

$$\int_C \tau\, ds = 0, \tag{26}$$

where τ denotes the torsion of the curve C in E^3 and s the arc length of C.

In the remaining part of this paper, we want to find analogous results for higher dimensional submanifolds.

Theorem 7. *Let M be an oriented closed submanifold of E^m. Then M is a spherical submanifold in E^m when and only when for all $(n-1)$-dimensional oriented closed submanifolds N of M, the integral*

(27) $$\int_N \theta = 0,$$

where θ *is given by* (19) *and* n *is the dimension of* M.

Proof. If for all $(n-1)$-dimensional oriented closed submanifolds N of M, (27) holds. Then, for any n-dimensional bounded submanifold $\overline{M}$ of M, we have

$$\int_{\partial \overline{M}} \theta = 0.$$

Thus we get $\int_{\overline{M}} d\theta = 0$. Since this is true for all n-dimensional bounded submanifolds of M, we obtain $d\theta = 0$. Therefore, by (20), we get $\boldsymbol{X}\cdot\boldsymbol{H} = -1$. Thus, by Theorem 1 of [2], we see that M is a spherical submanifold of E^m. Conversely, if M is spherical, then we have $d\theta = 0$. Hence, for all $(n-1)$-dimensional oriented closed submanifolds N of M, (27) holds.

Similarly, for a hypersurface M in E^{n+1}, if we put

(28) $$\sigma = \sum (-1)^{i-1} \omega_{n+1,1} \wedge \cdots \wedge \hat{\omega}_{n+1,i} \wedge \cdots \wedge \omega_{n+1,n} (\boldsymbol{X}\cdot\boldsymbol{e}_i),$$

then σ is a well-defined $(n-1)$-form on M and we have

Theorem 8. *Let* M *be an oriented closed convex hypersurface of* E^{n+1}. *Then* M *is a hypersphere of* E^{n+1} *centered at the origin when and only when for all* $(n-1)$*-dimensional oriented closed submanifolds* N *of* M, *the integral*

(29) $$\int_N \sigma = 0.$$

Proof. Suppose that for all $(n-1)$-dimensional oriented closed submanifolds N of M, (29) holds. Then for any n-dimensional bounded submanifold $\overline{M}$ of M, we have $\int_{\partial \overline{M}} \sigma = 0$. Thus we obtain $\int_{\overline{M}} d\sigma = 0$. Hence we get $d\sigma = 0$. On the other hand, by taking exterior derivative of (28), we obtain $d\sigma = (-1)^{n-1} n (K_{n-1} + (\boldsymbol{X}\cdot\boldsymbol{e}) K_n) dV$. Thus we obtain

(30) $$K_{n-1} + (\boldsymbol{X}\cdot\boldsymbol{e}) K_n = 0.$$

On the other hand, by (20), we have

(31) $$\int_M (1 + (\boldsymbol{X}\cdot\boldsymbol{e}) K_1) dV = 0.$$

From (30) and (31) we obtain

$$\int_M \left(\frac{1}{K_n}\right)(K_1 K_{n-1} - K_n)dV = 0. \tag{32}$$

Since M is covex, K_i is of same sign, $i=1, \cdots, n$. Hence, by Newton's inequalities and (32), we obtain $K_1 K_{n-1} = K_n$. This implies that M is totally umbilical. Hence K_i are constants. By (30) we see that the support function $\boldsymbol{X}\cdot\boldsymbol{e}$ is a constant. By Corollary 1 to Theorem 3, we see that M is spherical. The converse of this is trivial.

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