ON THE SCALAR CURVATURE OF IMMERSED MANIFOLDS

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In §1, we introduce the notion of α-th scalar curvatures: $\lambda_1, \cdots, \lambda_N$, and find the relationship between the scalar curvature and the α-th scalar curvatures for an n-dimensional riemannian manifold isometrically immersed in a euclidean space E^{n+N} of dimension $n+N$. In §2, we use these new curvatures to find a characterization of hypersphere in higher dimensional euclidean space and find some inequalities. In the last section, we find an equality for submanifolds with vanishing scalar curvature which generalizes a result of Ōtsuki.

1. i-th total absolute curvatures and α-th scalar curvatures

Let M^n be an n-dimensional closed riemmannian manifold[1] with an isometric immersion $x : M^n \to E^{n+N}$. Let $F(M^n)$ and $F(E^{n+N})$ be the bundles of orthonormal frames of M^n and E^{n+N}, respectively. Let B be the set of elements $b=(p, e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+N})$ such that $(p, e_1, \cdots, e_n) \in F(M^n)$ and $(x(p), e_1, \cdots, e_{n+N}) \in F(E^{n+N})$ whose orientation is coherent with the one of E^{n+N}, identifying e_i with $dx(e_i)$, $i=1, 2, \cdots, n$. Then $B \to M^n$ may be considered as a principal bundle with fibre $O(n) \times SO(N)$, and $\tilde{x} : B \to F(E^{n+N})$ is naturally defined by $\tilde{x}(b)=(x(p), e_1, \cdots, e_{n+N})$. Let B_v be the set of unit normal vectors of M^n in E^{n+N}.

The structure equations of E^{n+N} are given by

$$(1)\qquad \begin{cases} dx=\sum\limits_A \omega'_A e_A, \quad de_A=\sum\limits_B \omega'_{AB} e_B, \\ d\omega'_A=\sum\limits_B \omega'_B \wedge \omega'_{BA}, \quad d\omega'_{AB}=\sum\limits_C \omega'_{AC} \wedge \omega'_{CB}, \quad \omega'_{AB}+\omega'_{BA}=0 \\ \qquad A, B, C, \cdots =1, 2, \cdots, n+N, \end{cases}$$

where ω'_A, ω'_{AB} are differential 1-forms on $F(E^{n+N})$. Let ω_A, ω_{AB} be the induced 1-forms on B from ω'_A, ω'_{AB} by the mapping $\tilde{x}$.

Then we have

1) Manifolds, mappings, metrics, $\cdots$etc. are assumed to be differentiable and of class C^∞. We restriet ourselves only to connected manifolds of dimenson $n \geq 2$.

$$(2)\qquad \begin{aligned}&\omega_\gamma=0,\ \ \omega_{\gamma i}=\sum_j A_{\gamma ij}\omega_j,\ \ A_{\gamma ij}=A_{\gamma ji}\\ &i, j, \cdots=1, 2, \cdots, n\ ;\quad \gamma, t, \cdots=n+1, \cdots, n+N.\end{aligned}$$

From (1) we get

$$(3)\qquad \begin{cases} d\omega_i=\sum_j \omega_j \wedge \omega_{ji}, \\ d\omega_{ij}=\sum_k \omega_{ik} \wedge \omega_{kj}+\frac{1}{2}\sum_{k,h} R_{ijkh}\,\omega_k\wedge\omega_h, \\ R_{ijkh}=\sum_\gamma A_{\gamma ih}A_{\gamma jk}-\sum_\gamma A_{\gamma ik}A_{\gamma jk}. \end{cases}$$

The volume element of M^n can be written as

$$(4)\qquad dV=\omega_1\wedge\cdots\wedge\omega_n.$$

The $(N-1)$-form

$$(5)\qquad d\sigma_{N-1}=\omega_{n+N,n+1}\wedge\cdots\wedge\omega_{n+N,n+N-1}$$

can be regarded as an $(N-1)$-form on B_v. The $(n+N-1)$-form $dV\wedge d\sigma_{N-1}$ can be regarded as the volume element of B_v.

Let $A_{\gamma ij}$; $\gamma=n+1, \cdots, n+N, i, j=1, 2, \cdots, n,$ be given as in (2). The symmetric matrix $(A_{\gamma ij})$, $\gamma=n+1, \cdots, n+N$, is called the second fundamental form at (p, e_γ). we define the h-th mean curvature $K_h(p, e_\gamma)$ at $(p, e_\gamma)\in B_v$ by

$$(6)\qquad \det\,(\delta_{ij}+tA_{\gamma ij})=1+\sum_h \binom{n}{h} K_h(p, e_\gamma)t^h,$$

where δ_{ij} is the Kronecker delta, t a parameter and $\binom{n}{h}=n!/h!(n-h)!$.

We call the integral

$$(7)\qquad K_i^*(p)=\int |K_i(p, e)|^{\frac{n}{i}}d\sigma_{N-1}$$

over the sphere of unit normal vectors at $x(p)$, the i-th total absolute curvature of the immersion x at p, and we define as the *i-th total absolute curvature* of M^n itself the integral $\int_{M^n} K_i^*(p)dV$.

In the following, the *geodesic codimension* of M^n in E^{n+N} means the minimum of codimensions of M^n in linear subspaces of E^{n+N} containng M^n.

In [3, 4], the author proved that

Theorem A. *Let $x: M^n \to E^{n+N}$ be an immersion of an n-dimensional closed manifold M^n into E^{n+N}. Then we have*

$$\int_{M^n} K_i^*(p)\, dV \geqq 2c_{n+N-1}, \quad i=1, 2, \cdots, n, \tag{8}$$

where c_{n+N-1} is the area of unit $(n+N-1)$-sphere. The equality of (8) *holds when and only when M^n is imbedded as a hypersphere if $i<n$ and as a convex hypersurface if $i=n$ in an $(n+1)$-dimensional linear subspace of E^{n+N}.*

Remark 1.1. If $i=n$, then Theorem A is the well-known Chern-Lashof's theorem [5]. In this case; the inequality (8) can be improved as

$$\int_{M^n} K_i^*(p)\, dV \geqq \sum_{i=0}^{n} \beta_i(M^n)\, c_{n+N-1}, \tag{9}$$

where $\beta_i\,(M^n)$ is the i-th betti number of M^n.

To describe how $K_2\,(p, e)$ depends on e, we take a local cross-section of M^n in B, described by the function $\tilde{e}_A(q)$ for q in a neighborhood of p. Then for any frame $e_A(q)$ in B at $x(q)$, we have $e_{n+N}=\sum \xi_\gamma \tilde{e}_\gamma(q)$ and

$$A_{n+Nij}=\sum_\gamma \xi_\gamma \tilde{A}_{\gamma ij}, \quad \sum_\gamma \xi_\gamma^2=1, \tag{10}$$

where $\tilde{A}_{\gamma ij}$ is the function $A_{\gamma ij}$ restricted to the local cross-section. By (6), we find that $K_2(p, e_\gamma)$ is given by

$$\binom{n}{2} K_2(p, e_\gamma)=\sum_{i<j} (A_{\gamma ii}A_{\gamma jj}-A_{\gamma ij}^2). \tag{11}$$

Combining (10) and (11), we find that

$$\binom{n}{2} K_2(p, e_{n+N})=\sum_{i<j} [(\sum_\gamma \xi_\gamma A_{\gamma ii})(\sum_s \xi_s \tilde{A}_{sjj})-(\sum_t \xi_t \tilde{A}_{tij})^2]. \tag{12}$$

The right hand side is a quadratic form of $\xi_{n+1}, \cdots, \xi_{n+N}$. Hence, choosing a suitable cross-section, we can write $K_2(p, e_{n+N})$ as

$$K_2(p, e_{n+N})=\sum_\gamma \lambda_{\gamma-n}(p)\xi_\gamma\xi_\gamma, \quad \lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_N. \tag{13}$$

We may call this local cross-section of $B\to F(M^n)$ a *Frenet cross-section* and such frame $(p, e_1, \cdots, e_n, \bar{e}_{n+1}, \cdots, \bar{e}_{n+N})$ a *Frenet frame*. We call λ_α; $\alpha = 1, \cdots, N$, the α-th scalar curvature of M^n in E^{n+N}. By means of the method of definition, λ_α is defined continuously on the whole manifold M^n and is differentiable on the open subset in which $\lambda_1 > \lambda_2 > \cdots > \lambda_N$. Now, let $\tilde{e}_i = e_i$, and $\tilde{e}_\gamma = \bar{e}_\gamma$, where $(p, e_1, \cdots, e_n, \bar{e}_{n+1}, \cdots, \bar{e}_{n+N})$ is the Frenet frame, then by (12) and (13), we have

$$(14) \qquad \binom{n}{2} \lambda_{\gamma-n}(p) = \sum_{i<j} (\tilde{A}_{\gamma ii} \tilde{A}_{\gamma jj} - \tilde{A}^2_{\gamma ij}).$$

Hence, we get

$$(15) \qquad \binom{n}{2} \sum_\gamma \lambda_{\gamma-n}(p) = \sum_{i<j} (\sum_\gamma \tilde{A}_{\gamma ii} \tilde{A}_{\gamma jj} - \tilde{A}^2_{\gamma ij}) = \sum_{i<j} R_{ijji}.$$

Thus, if we define the scalar curvature $S(p)$ as

$$(16) \qquad \binom{n}{2} S(p) = \sum_{i<j} R_{ijji}$$

Then $S(p)$ is intrinsic and we have

Proposition 1.1. *The scalar curvature $S(p)$ and the α-th scalar curvature satisfy the following relation*:

$$(17) \qquad S(p) = \lambda_1(p) + \cdots + \lambda_N(p).$$

2. Some results on α-th scalar curvature

Now, let us define an invariant by

$$(18) \qquad \rho(p) = \max\{\sqrt{|\lambda_1(p)|^n}, \cdots, \sqrt{|\lambda_N(p)|^n}\}$$
$$= \max\{\sqrt{|\lambda_1(p)|^n}, \sqrt{|\lambda_N(p)|^n}\}.$$

Then, from (7) and (13), we have

$$(19) \qquad K_2^*(p) = \int |K_2(p, e_{n+N})|^{\frac{n}{2}} d\sigma_{N-1} = \int |\sum \lambda_{\gamma-n}(p) \xi_\gamma \xi_\gamma|^{\frac{n}{2}} d\sigma_{N-1}$$
$$\leqq \rho(p) \int (\sum \xi_\gamma \xi_\gamma)^{\frac{n}{2}} d\sigma_{N-1} = \rho(p) c_{N-1}.$$

Accordingly, from Theorem A and (19), we get

$$(20) \qquad c_{N-1} \int_{M^n} \rho(p)\, dV \geqq 2 c_{n+N-1}.$$

If the equality of (20) holds, then by (19) we get

$$\int_{M^n} K_2^* \, dV = 2c_{n+N-1}.$$

Hence, by Theorem A, we know that M^n is imbedded as a hypersphere if $n>2$ and as a convex hypersurface if $n=2$. Therefore, we can easily derive that

$$\lambda_1(p) \geqq 0, \quad \lambda_2(p) = \cdots = \lambda_N(p) = 0, \quad \text{for all } p \in M^n.$$

Thus, by the equalities of (19) and (20), we find that the geodesic codimension is equal to 1. Hence, we have proved the following theorem:

Theorem 2.1. *For an n-dimensional closed manifold M^n immersed in E^{n+N}, we have*

$$\int_{M^n} \rho(p) dV \geqq \frac{2c_{n+N-1}}{c_{N-1}}. \tag{21}$$

The equality of (21) *holds when and only when M^n is imbedded as a hypersphere if $n>2$, and as a convex hypersurface if $n=2$ in an $(n+1)$-dimensional linear subspace of E^{n+N}.*

From Theorem 2. 1, we have the following corollaries:

Corollary 2.1. *Let M^n be an n-dimensional riemannian closed manifold. Then M^n can not be isometrically immersed in a euclidean space E^{n+N} with $\rho(p) \leqq \dfrac{2c_{n+N-1}}{c_{N-1}\, v(M^n)}$ except $\rho(p) = constant$, $N=1$ and M^n is an n-sphere, where $v(M^n)$ denotes the volume of M^n.*

Corollary 2.2. *Let M^n be an n-dimensional riemannian closed manifold. If M^n is not an n-sphere and the scalar curvature $S(p)$ of M^n satisfies*

$$\int_{M^n} \sqrt{|S(p)|^n} \, dV \leqq c_n, \tag{22}$$

then M^n can not be isometrically immersed in E^{n+1}.

Remark 2.1. Flat torus in E^4 is an example that Corollary 2. 2 is not true in general if $N \geqq 2$.

Remark 2.2. If M^n is isometrically immersed in E^{n+1} as a closed hypersurface, then for any $p \in M^n$ such that p is a nondegenerate critical point of a height function in the direction e. Then the second fundamental form at (p, e) is definite. Thus the scalar curvature $S(p)$ is positive. This shows that every n-dimensional closed riemannian manifold M^n isometrically immersed in E^{n+1} must have positive scalar curvature $S(p)$ at some points.

Furthermore, we can use the α-th scalar curvature to characterize the hypersphere as follows:

Theorem 2.2. *Let M^n $(n \geqq 3)$ be an n-dimensional closed manifold immersed in E^{n+N}. Then*

$$(23) \qquad \int_{M^n} (\lambda_1)^{\frac{1}{2}n}\, dV = c_n \qquad \text{and} \quad \lambda_2 = \cdots = \lambda_N = 0$$

if and only if M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+N}.

Proof. Assume that (23) holds, then, by (13), we have

$$K_2(p, e_{n+N}) = \lambda_1 \xi_1^2 \geqq 0.$$

Thus we get

$$K_2^*(p) = \int \sqrt{(\lambda_1 \xi_1^2)^n}\, d\sigma_{N-1} = (\lambda_1)^{\frac{1}{2}n} \int |\xi_1|^n\, d\sigma_{N-1}$$
$$= \frac{2c_{n+N-1}}{c_n} (\lambda_1)^{\frac{1}{2}n}.$$

By integrating both sides of the above formula, we get

$$\int_{M^n} K_2^*(p) dV = \frac{2c_{n+N-1}}{c_n} \int_{M^n} (\lambda_1)^{\frac{1}{2}n}\, dV = 2c_{n+N-1}.$$

Thus, we know that M^n is imbedded as a hypersphere in an $(n+1)$-dimensional linear subspace of E^{n+N}. The converse of this is trivial.

Remark 2.3. If $n=2$, then Theorem 2.2 was treated in [1].

3. Submanifolds with $S(p)=0$

The main object of this section is to prove the following theorem:

Theorem 3.1. *Let M^{2m} be a 2m-dimensional closed manifold immersed in E^{2m+2} with scalar curvature $S(p)=0$. Then we have*

$$\int_{M^{2m}} (\lambda_1)^m \, dV = \frac{c_m}{2c_{m+1}} \int_{M^{2m}} K_2^* \, dV. \tag{24}$$

Proof. By the assumption, we have

$$\lambda_1(p) + \lambda_2(p) = S(p) = 0.$$

Hence, we have

$$K_2^*(p) = \int_0^{2\pi} |K_2(p, e_{2m+2})|^m \, d\theta = (\lambda_1)^m \int_0^{2\pi} |\cos^2\theta - \sin^2\theta|^m \, d\theta$$
$$= (\lambda_1)^m \int_0^{2\pi} |\cos 2\theta|^m \, d\theta = \frac{2c_{m+1}}{c_m}(\lambda_1)^m.$$

Therefore, by integrating both sides of the above formula, we get (24).

In the special case, $m=1$, Theorem 3.1 reduces to a theorem of Ōtsuki [7]:

Theorem 3.2. *Let M^2 be a flat torus immersed in E^4. Then*

$$\int_{M^2} \lambda_1(p) dV \geqq 2\pi^2. \tag{25}$$

Equality holds if and only if $\int_{M^2} K_2^* \, dV = 8\pi^2$.

Theorem 3.2 follows immediately from Theorem 3.1 and (9).

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