ON A GENERALIZATION OF SEMI-INNER PRODUCT SPACES

B. NATH

1. In his paper entitled 'Semi-Inner Product Spaces' Lumer has considered vector spaces on which instead of a bilinear form there is defined a form $[x, y]$ which is linear in one component only, strictly positive and satisfies a Schwarz's inequality. He calls this space a semi-inner product space. In the present paper we give a straight forward generalization of a semi-inner product space by replacing a Schwarz's inequality by a Hölder's inequality. We show that a generalized semi-inner product induces a norm by setting $\|x\| = ([x, x])^{1/p}$, $1<p<\infty$; and for every normed space we can construct a generalized semi-inner product space. For $p=2$, this theorem reduces to Theorem 2 of Lumer [1, p. 31].

2. Definition. Let X be a vector space over the scalar field $\mathscr{F}$, where $\mathscr{F}$ is the fiield of real or complex numbers. Consider a functional defined on $X\times X$ as follows:

$$X\times X \longrightarrow \mathscr{F}$$
$$\langle x, y\rangle \longrightarrow [x, y].$$

If $[x, y]$ satisfies the postulates:

(1) $[x+y, z]=[x, z]+[y, z]$, x, y and $z\in X$,

(2) $[\lambda x, y]=\lambda[x, y]$, $\lambda\in\mathscr{F}$ and $x, y\in X$,

(3) $[x, x]>0$ for $x\neq 0$,

(4) $|[x, y]|\leq [x, x]^{\frac{1}{p}}[y, y]^{\frac{p-1}{p}}$, $1<p<\infty$,

then, we say that $[x, y]$ is a *generalized semi-inner product* on X.

A vector space X, together with a generalized semi-inner product defined on it, will be called a generalized semi-inner product space which may be abbreviated as g. s. i. p. s. For $p=2$, a generalized semi-inner product space becomes a semi-inner product space.

3. 1. We prove the following theorem:

Theorem. *A generalized semi-inner product space is a normed linear* *space with the norm*

$$[x, x]^{\frac{1}{p}}, \quad 1<p<\infty.$$

Every normed linear space can be made into a generalized semi-inner product space.

3. 2. For the proof of the theorem we shall be giving the following three auxiliary results in the form of Lemmas I, II and III.

Lemma I. *A generalized semi-inner product space is a normed linear space with the norm* $[x, x]^{\frac{1}{p}}$.

Proof. It is to be shown that $\|x\|=[x, x]^{\frac{1}{p}}$ is a norm. For this here it is sufficient to prove that

(i) $[\alpha x, \alpha x]^{\frac{1}{p}}=|\alpha|\,[x, x]^{\frac{1}{p}}$ and (ii) $[x+y, x+y]^{\frac{1}{p}}\leqq[x, x]^{\frac{1}{p}}+[y, y]^{\frac{1}{p}}$.

Proof of (i). From postulate (2) of a g. s. i. p., we have

$$[\alpha x, \alpha x]=\alpha[x, \alpha x].$$

Therefore,

$$|[\alpha x, \alpha x]|=|\alpha|\,|[x, \alpha x]|.$$

From postulate (3) of a g. s. i. p.,

$$|[\alpha x, \alpha x]|=[\alpha x, \alpha x]$$

and accordingly

$$[\alpha x, \alpha x]=|\alpha|\,|[x, \alpha x]|.$$

Using postulate (4) of a g. s. i. p., we get

$$|[x, \alpha x]|\leqq[x, x]^{\frac{1}{p}}[\alpha x, \alpha x]^{\frac{p-1}{p}}, \quad 1<p<\infty.$$

Therefore

$$[\alpha x, \alpha x]\leqq|\alpha|\,[x, x]^{\frac{1}{p}}[\alpha x, \alpha x]^{\frac{p-1}{p}}.$$

Hence, we obtain

(1. 1) $$[\alpha x, \alpha x]^{\frac{1}{p}}\leqq|\alpha|\,[x, x]^{\frac{1}{p}}.$$

Since we can write

$$[x, x]^{\frac{1}{p}}=\left[\frac{1}{\alpha}\alpha x, \frac{1}{\alpha}\alpha x\right]^{\frac{1}{p}} \text{ for } \alpha\neq 0,$$

it follows from (1. 1) that

$$[x, x]^{\frac{1}{p}} \leqq \frac{1}{|\alpha|}[\alpha x, \alpha x]^{\frac{1}{p}}.$$

Therefore

(1. 2) $$|\alpha|[x, x]^{\frac{1}{p}} \leqq [\alpha x, \alpha x]^{\frac{1}{p}} \text{ for all } \alpha \in \mathscr{F}$$

Combining (1. 1) and (1. 2), we have

$$[\alpha x, \alpha x]^{\frac{1}{p}} = |\alpha|[x, x]^{\frac{1}{p}}.$$

Proof of (ii). From postulate (1) of a g. s. i. p., we have

$$[x+y, x+y] = [x, x+y] + [y, x+y].$$

By virtue of postulate (2) of a g. s. i. p., we get

(1. 3) $$[x+y, x+y] = |[x, x+y] + [y, x+y]| \leqq |[x, x+y]| + |[y, x+y]|.$$

Also, by postulate (4) of a g. s. i. p., we obtain

(1. 4) $$|[x, x+y]| \leqq [x, x]^{\frac{1}{p}}[x+y, x+y]^{\frac{p-1}{p}}, \quad 1<p<\infty.$$

Simillarly, we have

(1. 5) $$|[y, x+y]| \leqq [y, y]^{\frac{1}{p}}[x+y, x+y]^{\frac{p-1}{p}}, \quad 1<p<\infty.$$

From (1. 3), (1. 4) and (1. 5), we have

$$[x+y, x+y] \leqq \left\{[x, x]^{\frac{1}{p}} + [y, y]^{\frac{1}{p}}\right\}[x+y, x+y]^{\frac{p-1}{p}}.$$

Therefore, we obtain

$$[x+y, x+y]^{\frac{1}{p}} \leqq [x, x]^{\frac{1}{p}} + [y, y]^{\frac{1}{p}}.$$

Lemma II. *Let* x_0 *be a nonzero vector in the normed linear space* X. *Then there exists a bounded linear functional* F, *defined on the whole space, such that* $\|F\| = \|x_0\|^{p-1}$ *and* $F(x_0) = \|x_0\|^p$, *where* $1 \leqq p < \infty$.

Proof. Consider the sub-space

$$M = [\{x_0\}]$$

consisting of all scalar multiples of x_0. Consider the functional f defined on M as follows :

$$f : M \longrightarrow \mathscr{F} \text{ (the scalar field)}$$
$$\alpha x_0 \longrightarrow \alpha\|x_0\|^p.$$

We shall prove that f is a linear functional with the property

$$f(x_0)=\|x_0\|^p.$$

For all $x, y \in M$, we have

(2. 1) $$f(x+y)=f(\alpha x_0+\beta x_0)=(\alpha+\beta)\|x_0\|^p=f(x)+f(y)$$

and

(2. 2) $$f(\beta x)=f(\beta\alpha x_0)=\beta\alpha\|x_0\|^p=\beta f(x).$$

Combining (2. 1) and (2. 2), the linearity of f is proved.

By definition of f, we have

$$f(\alpha x_0)=\alpha\|x_0\|^p.$$

Taking $\alpha=1$, we get

(2. 3) $$f(x_0)=\|x_0\|^p.$$

Further, since for any $x\in M$,

(2. 4) $$|f(x)|=|f(\alpha x_0)|=|\alpha|\,\|x_0\|^p=\|x_0\|^{p-1}\|\alpha x_0\|=\|x_0\|^{p-1}\|x\|,$$

we see that f is a bounded linear functional.

Since f is a bounded linear functional,

$$\|f\|=\inf\{K: |f(x)|\leqq K\|x\| \text{ for all } x\in M\}.$$

From (2. 4) it is clear that

$$\|x_0\|^{p-1}\in\{K: |f(x)|\leqq K\|x\| \text{ for all } x\in M\}.$$

Therefore,

(2. 5) $$\|f\|\leqq\|x_0\|^{p-1}.$$

Since f is a bounded linear functional,

$$|f(x)|\leqq\|f\|\,\|x\| \text{ for all } x\in M.$$

From (2. 4), we have

$$\|x_0\|^{p-1}\|x\|\leqq\|f\|\,\|x\|.$$

Thus

(2. 6) $$\|x_0\|^{p-1}\leqq\|f\|.$$

From (2. 5) and (2. 6), we have

(2. 7) $$\|f\|=\|x_0\|^{p-1}.$$

Therefore by the Hahn-Banach theorem [2, Theorem 4. 3-A], we can extend f to a bounded linear functional F, defined on all of X such that $\|f\|=\|F\|$ and $f(x)=F(x)$ for all $x \in M$.

Since $\|f\|=\|F\|$, it follows from (2. 7) that

$$\|F\|=\|x_0\|^{p-1}.$$

Since $f(x)=F(x)$ for all $x \in M$, we obtain from (2. 3) that

$$F(x_0)=\|x_0\|^p.$$

Lemma III. *Every normed linear space can be made into a generalized semi-inner product space.*

Proof. Let X be a normed linear space. By the Lemma II for $x \in X$, there exists a bounded linear functional W_x such that $W_x(x)=\|x\|^p$ and $\|W_x\|=\|x\|^{p-1}$, where $1 \leqq p < \infty$. We proceed to verify that $[x, y]=W_y(x)$ defines a generalized semi-inner product. We claim that the following assertions about $W_y(x)$ are valid:

(1′) $W_z(x+y)=W_z(x)+W_z(y)$

(2′) $W_y(\lambda x)=\lambda W_y(x)$

(3′) $W_x(x)>0$ for $x \neq 0$

(4′) $|W_y(x)| \leqq \{W_x(x)\}^{\frac{1}{p}} . \{W_y(y)\}^{\frac{p-1}{p}}$.

Since W_z is a linear functional, (1′) and (2′) follow.
Since $W_x(x)=\|x\|^p$, we have $W_x(x)>0$ for $x \neq 0$ and (3′) holds.
Since W_y is a bounded linear functional,

$$|W_y(x)| \leqq \|W_y\| \, \|x\| . \tag{3. 1}$$

Since $\|W_y\|=\|y\|^{p-1}$ and $W_y(y)=\|y\|^p$, we have from (3. 1) that

$$|W_y(x)| \leqq \{W_x(x)\}^{\frac{1}{p}}\{W_y(y)\}^{\frac{p-1}{p}}.$$

This proves (5′).

3. 3. Proof of Theorem. The proof of the main theorem follows from Lemas I and III.

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