ON THE DIFFERENCE CURVATURE OF SURFACES IN EUCLIDEAN SPACE

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We consider a closed oriented surface M^2 and an immersion $x: M^2 \to E^{2+N}$ into the euclidean space E^{2+N} of dimension $2+N$. Let B_v be the bundle of unit normal vectors of $x(M^2)$, so that a point of B_v is a pair (p, e), e is a unit normal veotor to $x(M^2)$ at $x(p)$. Then to $(p, e) \in B_v$, we put

$$(1) \qquad I = dx \cdot dx, \qquad II_e = dx \cdot de,$$

where dx, de are vector-valued linear differential forms in B_v and "." is the scalar product. I and II_e are called the first and the second quadratic forms in B_v. The eigenvalues $k_1(p, e)$ and $k_2(p, e)$ of II_e relative to I are called the principal curvatures at (p, e). We call

$$(2) \qquad S(p, e) = \frac{1}{4}(k_1(p, e) - k_2(p, e))^2$$

the *difference curvature* of the immersion x at (p, e).

The main purpose of this paper is to get some global theorems for the difference curvature $S(p, e)$.

1. Preliminaries

Let M^2 be an oriented closed surface with an immersion $x: M^2 \to E^{2+N}$. Let $F(M^2)$ and $F(E^{2+N})$ be the bundles of orthonormal frames of M^2 and E^{2+N} respectively. Let B be the set of elements $b = (p, e_1, e_2, e_2, \cdots, e_{2+N})$ so that $(p, e_1, e_2) \in F(M^2)$ and $(x(p), e_1, \cdots, e_{2+N}) \in F(E^{2+N})$ whose orientation is coherent with the one of E^{2+N}, identifying e_i with $dx(e_i)$, $i=1$, 2. Then $B \to M^2$ may be considered as a principal bundle with fibre $SO(2) \times SO(N)$, and $\tilde{x}: B \to F(E^{2+N})$ is naturally defined by $\tilde{x}(b) = (x(p), e_1, \cdots, e_{2+N})$.

The structure equations of E^{2+N} are given by

$$dx = \sum_B \theta_A e_A, \qquad de_A = \sum_B \theta_{AB} e_B$$

$$(3) \qquad d\theta_A = \sum_B \theta_B \wedge \theta_{BA}, \qquad d\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB}, \qquad \theta_{AB} + \theta_{BA} = 0,$$

$$A, B, C, \cdots, = 1, 2, \cdots, 2+N,$$

where θ_A, θ_{AB} are differential 1-forms on $F(E^{2+N})$. Let ω_A, ω_{AB} be the induced 1-forms on B from θ_A, θ_{AB} by the mapping $\tilde{x}$. Then we have

$$(4) \qquad \omega_r=0, \qquad \omega_{ir}=\sum_j A_{rij}\omega_j, \qquad A_{rij}=A_{rji},$$
$$i, j, k, \dots=1, 2\,; \qquad r, s, t, \dots=3, \dots, 2+N.$$

The principal curvatures $k_1(p, e)$ and $k_2(p, e)$ are the eigenvalues of the matrix (A_{rij}) where $e=e_r$. The mean curvature $K_1(p, e)$ and the total curvature $K_2(p, e)$ are given by

$$(5) \qquad K_1(p, e)=\frac{1}{2}(k_1(p, e)+k_2(p, e)), \qquad K_2(p, e)=k_2(p, e)k_2(p, e).$$

Hence we have

$$(6) \qquad K_1(p, e)=\frac{1}{2}\mathrm{trace}(A_{rij}), \qquad K_2(p, e)=\det(A_{rij}).$$

Let dV be the volume element of M^2. There is a differential form $d\sigma$ of degree $N-1$ on B_v such that its restriction to a fibre is the volume element of the sphere of unit normal vectors at a point $p\in M^2$; then $d\sigma\wedge dV$ is the volume element of B_v. In fact, we have

$$(7) \qquad dV=\omega_1\wedge\omega_2, \qquad d\sigma=\omega_{2+N,3}\wedge\cdots\wedge\omega_{2+N,1+N}.$$

We call the integral

$$(8) \qquad S^*(p)=\int S(p, e)\, d\sigma$$

over the sphere of unit normal vectors at $x(p)$ the *difference curvature of* M^2 *at* p, and define as the *difference curvature of the immersion* x itself $\int_{M^2} S^*(p)dV$, if it converges.

2. Some Global Theorems of the Difference Curvature

In [1] and [2], we have proved the following:

Lemma 1. *Let M^2 be an oriented closed surface immersed in E^{2+N}. If*

$$(9) \qquad \int_{B_v} K_1(p, e)^2 dV\wedge d\sigma=2c_{N+1},$$

then M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N}, where c_{N+1} denotes the area of the unit $(N+1)$-sphere.

The main aim of this section is to prove the following:

Theorem 1. *Let M^2 be an oriented closed surface immersed in E^{2+N}. Its difference curvature satisfies the inequality:*

(10) $$\int_{M^2} S^*(p)dV \geqq 2gc_{N+1},$$

where g is the genus of M^2. *The equality sign holds when and only when* M^2 *is imbedded as a sphere in a 3-dimensional linear subspace of* E^{2+N}.

Proof. Let M^2 be an oriented closed surface immersed in E^{2+N}. Let $(p, e_1, e_2, \bar{e}_3, \cdots, \bar{e}_{2+N})$ be a local cross-section of $B \to F(M^2)$ defined on U and for any e in S_p^{N-1}, $p \in U$, put $e = e_{2+N} = \sum_r \xi_r \bar{e}_r(p)$. Denoting the restriction of A_{rij} onto the image of this local cross-section by $\bar{A}_{rij}$. We may put

$$A_{2+Nij} = \sum_r \xi_r \bar{A}_{rij}.$$

From (6) we get

(11) $$K_2(p, e) = \det(\sum_r \xi_r \bar{A}_{rij}) = (\sum_r \xi_r \bar{A}_{r11})(\sum_s \xi_s \bar{A}_{s22}) - (\sum_t \xi_t \bar{A}_{t12})^2.$$

The right hand side is a quadratic form of $\xi_3, \cdots, \xi_{2+N}$. Hence, by choosing a suitable cross-section, we can write $K_2(p, e)$ as

(12) $$K_2(p, e) = \sum \lambda_{r-2} \xi_r \xi_r, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N.$$

Thus, by (12), we have

(13) $$\int_{B_v} K_2(p, e) dV \wedge d\sigma = \int_{B_v} (\lambda_1(p)\cos^2\theta_1 + \cdots + \lambda_N(p)\cos^2\theta_N) dV \wedge d\sigma$$
$$= \frac{c_{N+1}}{2\pi} \int_{M^2} (\lambda_1(p) + \cdots + \lambda_N(p)) dV.$$

On the other hand, we have [4]

(14) $$G(p) = \lambda_1(p) + \cdots + \lambda_N(p),$$

where $G(p)$ denotes the Gaussian curvature of M^2 at p. Hence, by the Gauss-Bonnet theorem, we have

(15) $$\int_{B_v} |K_2(p, e)| dV \wedge d\sigma = (2-2g)c_{N+1}.$$

On the other hand, by an inequality of Chern-Lashof [3], we have

(16) $$\int_{B_v} |K_2(p, e)| dV \wedge d\sigma \geq (2+2g)c_{N+1}.$$

Therefore, if we put

(17) $$V+ = \{(p, e) \in B_v : K_2(p, e) \geq 0\}, \quad V- = \{(p, e) \in B_v : K_2(p, e) < 0\}.$$

Then, by (15) and (16), we have

$$(18) \qquad -\int_{V-} K_2(p, e)dV \wedge d\sigma \geqq 2gc_{N+1}.$$

Therefore, by (2), (5) and (18), we get

$$(19) \qquad \int_M S^*(p)dV = \int_{B_\nu} S(p, e)dV \wedge d\sigma \geqq \int_{V-} S(p, e)dV \wedge d\sigma$$
$$= \int_{V-} (K_1(p, e)^2 - K_2(p, e))dV \wedge d\sigma$$
$$\geqq \int_{V-} -K_2(p, e)dV \wedge d\sigma \geqq 2gc_{N+1}.$$

This proves (10). Now suppose the equality of (10) holds, then, by (19), we get

$$(20) \qquad -\int_{V-} K_2(p, e)dV \wedge d\sigma = 2gc_{N+1},$$

and

$$(21) \qquad S(p, e) = 0 \quad \text{on } V+, \qquad \text{and} \quad K_1(p, e) = 0 \qquad \text{on } V-.$$

By (15) and (20), we have

$$(22) \qquad \int_{V+} K_2(p, e)dV \wedge d\sigma = 2c_{N+1}.$$

Therefore, by (2), (5), (21) and (22), we have

$$(23) \qquad \int_{B_\nu} K_1(p, e)^2 dV \wedge d\sigma = \int_{V+} K_1(p, e)^2 dV \wedge d\sigma = \int_{V+} K_2(p, e)dV \wedge d\sigma$$
$$= 2c_{N+1}.$$

Thus, by Lemma 1 and (23), we know that M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N}. Conversely, if M^2 is imbedded as a sphere in a 3-dimensional linear subspace of E^{2+N}, then it is easy to verify that the equality of (10) holds. This completes the proof of the theorem.

From theorem 1 we can easily get the following :

Theorem 2. *Let M^2 be an oriented closed surface immersed in E^{2+N}. If the difference curvature satisfies the inequality :*

$$(24) \qquad \int_{M^2} S^*(p)dV \leqq 2c_{N+1},$$

then M^2 is diffeomorphic to a sphere.

Theorem 3. *Let* M^2 *be an oriented closed surface immersed in* E^{2+N}. *If the difference curvature satisfies the inequality* :

$$\int_{M^2} S^*(p)dV \leqq 4c_{N+1}, \tag{25}$$

then M^2 *is either diffeomorphic to a sphere or a torus.*

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