ON SEPARABLE EXTENSIONS OF DOMAINS

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Throughout this paper we shall understand by a domain a commutative ring with identity element and without zero divisors $\neq 0$, and all rings will be assumed commutative with identity element. Let B be a domain. If the quotient field of B contains no finitely generated separable B-subalgebras containing B but not coinciding with B then we say that B is *a separably closed domain*. Clearly any integrally closed domain is separably closed. In Th. 1, we see that B is a separably closed domain if and only if every finitely generated, separable B-algebra containing B but not containing proper idempotents (i. e., not containing idempotents except 0 and 1) is a domain which is projective over B. In virtue of the result, we prove some structure theorems of separable algebras over a separably closed domain, containing a Janusz's result [6, Th. 4. 3] which is a structure theorem of finitely generated separable algebras over a Noetherian integrally closed domain.

In what follows, we shall summarize the notations and definitions which will used in the subsequent study. For a ring S and for a subring R, we denote by $\mathfrak{J}(S/R)$ the group of (ring) automorphisms in S which leave the elements of R fixed, and moreover, for a set $\mathfrak{H}$ of (ring) automorphisms in S, we denote by $\mathfrak{H}|R$ the restriction of $\mathfrak{H}$ to R and denote by $J(\mathfrak{H})$ (resp. by $J(\mathfrak{H}|R)$) the fixring of $\mathfrak{H}$ in S (resp. $J(\mathfrak{H}) \cap R$).

As in [6] and [8], an R-algebra S will be called respectively

f. g. separable, if S is finitely generated and separable over R;

locally f. g. separable, if every finite set of elements in S is contained in an f. g. separable R-subalgebra of S;

strongly separable, if S is finitely generated, projective, and separable over R;

locally strongly separable, if every finite set of elements in S is contained in a strongly separable R-subalgebra of S.

The property of being f. g. separable (resp. strongly separable) is transitive (see the proof of [2, Th. 2. 3]). From this, it follows easily that the property of being locally f. g. separable (resp. locally strongly separable) is transitive.

Now our study starts with the following

Proposition 1. *Let A be a ring without proper idempotents, and $R \subset S \subset T$ subrings of A. If T is f. g. separable over R and $J(\mathfrak{J}(A/S))=S$ then*

(1) *T is strongly separable over S and $J(\mathfrak{J}(A/T))=T$,*

(2) *S is f. g. separable over R.*

Proof. (1) is contained [8, Th. 1], and the argument used in the proof of [2, Th. 2. 3] enables us to obtain the remaining.

Corollary 1. *Let A be a ring without proper idempotents, and $R\subset S\subset T$ subrings of A. If T is locally f. g. separable over R and $J(\mathfrak{J}(A/S))=S$ then*

(1) *T is locally strongly separable over S, and*

(2) *S is locally f. g. separable over R.*

Proof. Obviously T is locally f. g. separable over S. Then by Prop. 1, T is locally strongly separable over S. Let F be a finite subset of S, and T' a subring of T containing $R[F]$ which is f. g. separable over R. Then, by making use of the same method as in the proof of [8, Th. 1], we can prove that $\mathfrak{J}(A/S)\,|\,T'$ is a finite set. Hence the subring A' of A generated by $\bigcup_{\sigma\in\mathfrak{J}(A/S)}\sigma(T')$ is an f. g. separable R-algebra, and $\mathfrak{J}(A/S)\,|\,A'$ is a group of automorphisms in A'. By Prop. 1, $J(\mathfrak{J}(A/S)\,|\,A')$ is an f. g. separable R-algebra which is a subring of S containing F. Thus S is locally f. g. separable over R.

As in [6], if R is a ring without proper idempotents then there exists a locally strongly separable R-algebra, Ω, such that Ω has no proper idempotents and if Γ is a strongly separable Ω-algebra with no proper idempotents then $\Gamma=\Omega$ ([6, Prop. 1. 4]). Such an R-algebra will be called *a separable closure of R*. The separable closure Ω of R is unique up to isomorphisms, and every locally strongly separable R-algebra without proper idempotents can be imbedded in Ω ([6, Prop. 1. 7]). Moreover there holds $J(\mathfrak{J}(\Omega/R))=R$ ([6, Prop. 1. 9]). Combining these results with Coro. 1 and [8, Th. 3], we can easily obtain the following

Proposition 2. *Let A be a ring without proper idempotents which contains subrings $R\subset S$, and let A be locally strongly separable over R. Then, A is locally strongly separable over S if and only if S is locally f. g. separable over R. If the conditions hold then S is locally* *strongly separable over* R.

Similarly we have the following

Corollary 2. *Let* A *be a ring without proper idempotents which contains subrings* $R\subset S$, *and let* A *be locally f. g. separable over* R, *and* $J(\mathfrak{J}(A/R))=R$. *Then* A *is locally strongly separable over* R, *and the following conditions are equivalent.*

(a) $J(\mathfrak{J}(A/S))=S$.

(b) A *is locally strongly separable over* S.

(c) S *is locally f. g. separable over* R.

(d) S *is locally strongly separable over* R.

Throughout the rest of this paper B will denote a domain, and we shall use the following conventions:

Ω: the separable closure of B.

Q: the quotient field of B.

$\bar{Q}$: the algebraic closure of Q.

$\bar{B}$: the integral closure of B in $\bar{Q}$ (i. e., the totality of elements of $\bar{Q}$ that are integral over B.)

Now we shall prove the following theorem which contains the results of [6, Lemma 4. 1 and Coro. 4. 2].

Theorem 1. *The following conditions are equivalent.*

(a) B *is a separably closed domain.*

(b) *Every domain containing* B *which is f. g. separable over* B *is projective over* B.

(c) *Every ring containing* B *which has no proper idempotents and is f. g. separable over* B *is a domain and is projective over* B.

Proof. We shall give a cyclic proof of this theorem in order: (a) implies (b), (b) implies (c), (c) implies (a). Assume (a). Let R be a domain containing B which is f. g. separable over B. Then $Q[R]$ is an f. g. separable field extension of Q. Hence $Q[R]$ is imbedded in a field K which is Galois over Q. Let S be the subring of K generated by $\bigcup_{\sigma\in\mathfrak{J}(K/Q)}\sigma(R)$. Then S is an f. g. separable extension of B and $\mathfrak{J}(K/Q)|S$ is a group of automorphisms of S. By Prop. 1, S is a strongly separable extension of $J(\mathfrak{J}(K/Q)|S)$ and $J(\mathfrak{J}(K/Q)|S)$ is an f. g. separable extension of B which is a subring of Q. Since B is a separably closed domain, it follows that $J(\mathfrak{J}(K/Q)|S)=B$; hence R is projective over B. Thus we have (b). Assume (b). Let R be an f. g.

separable ring extension of B without proper idempotents. Then $Q\otimes_B R$ is a direct sum of separable field extensions of Q ([9, Th. 1]). For the identity element e of one of the fields in this decomposition, R is B-algebra homomorphic to an f. g. separable domain $(1\otimes R)e$ containing B. By (b), $(1\otimes R)e$ is projective over B. Hence by [6, Lemma 1.6], we have a B-algebra isomorphism $R\cong(1\otimes R)e$. Thus we obtain (c). Assume (c). Let R be a subring of Q containing B which is f. g. separable over B. Then R is projective over B. Hence by [6, Th. 1.1], R is imbedded in a Galois extension S of B without proper idempotents. Then S is a domain too. Since Q is the quotient field of B and R is a subring of Q containing B, we have $R\subset J(\mathfrak{G}(S/B))=B$, and hence $R=B$. Therefore B is a separably closed domain. This completes the proof.

Remark 1. As in an example of G. J. Janusz ([6, P. 473]), we consider a ring $R=Z_{(2)}+Z_{(2)}\sqrt{5}$ where $Z_{(2)}$ is the localization at (2) in the ring of rational integers. Then $R[X]/(X^2+X-1)$ is a strongly separable R-algebra which has no proper idempotents and is not a domain. Hence by Th. 1, R is not separably closed.

Theorem 2. *B is a separably closed domain if and only if Ω is a domain and is separably closed.*

Proof. Let B be a separably closed domain. Then Ω is a domain by Th. 1 ((a)⇒(c)). Let Γ be a domain containing which is f. g. separable over Ω. Then Γ is locally f. g. separable over B. By Th. 1 ((a)⇒(b)), Γ is locally strongly separable over B. Hence $\Gamma\subset\Omega$, and this implies $\Gamma=\Omega$. Thus Ω is separably closed by Th. 1 ((b)⇒(a)). Assume that Ω is a domain and is separably closed. Let R be a domain containing B which is f. g. separable over B. Then $\Omega[R](\subset\bar{B})$ is f. g. separable over Ω. By Th. 1 ((a)⇒(b)), $\Omega[R]$ is projective over Ω. This gives $\Omega[R]=\Omega$, and so, R is projective over B by Prop. 2. Hence B is separably closed by Th. 1 ((b)⇒(a)).

Corollary 3. *Let B be a separably closed domain. Then every locally f. g. separable extension of B without proper idempotents is a separably closed domain which is a subring of Ω. In particular, every f. g. separable extension of B without proper idempotents is a separably closed domain which is projective over B.*

Proof. Let R be a locally f. g. separable extension of B without proper idempotents. Then by Th. 1, R is a domain and is locally strongly separable over B. Hence R is a subring of Ω, and by Prop. 2, Ω is the separable closure of R. Therefore by Th. 2, R is separably closed.

Corollary 4. *Let B be a separably closed domain, and R an intermediate ring of Ω/B. Then, R is a separably closed domain if and only if R is a locally f. g. separable B-algebra.*

Proof. Clearly Ω is locally f. g. separable over R. If R is a separably closed domain then, by Th. 1, Ω is locally strongly separable over R; then, by Prop. 2, R is locally f. g. separable over B. The converse part is a direct consequence of Coro. 3.

Combining Coro. 4 with [8, Th. 3], we have the following

Corollary 5. *Let B be a separably closed domain. Then there exists a $1-1$ dual correspondence between separably closed, intermediate rings of Ω/B and subgroups of $\mathfrak{J}(\Omega/B)$ which are closed in the finite topology, in the usual sense of Galois theory.*

Remark 2. Let B be a domain. Let $R\supset B$ and $S\supset B$ be subrings of $\bar{Q}$ such that R is a separably closed domain, and S is a locally f. g. separable B-algebra satisfying $J(\mathfrak{J}(S/B))=B$. Then $J(\mathfrak{J}(R[S]/R))=R$ and $J(\mathfrak{J}(S/R\cap S))=R\cap S$. Hence by Coro. 2, $S/R\cap S$ and $R\cap S/B$ are locally strongly separable extensions. Moreover, we have $R[S]\cong R\otimes_{R\cap S}S$ and $\mathfrak{J}(R[S]/R)\cong\mathfrak{J}(S/R\cap S)$. Next, let Ω' be the subring of $\bar{B}$ generated by all f. g. separable B-subalgebras, and set $B'=J(\mathfrak{J}(\Omega'/B))$. Then Ω' is locally f. g. separable over B and is the separable closure of B'; and B' is a subring of the quotient field of B which is separably closed and is locally f. g. separable over B.

Remark 3. Let B be a separably closed domain, and A an f. g. separable B-algebra which is B-torsion free. Then A is a direct sum of separably closed domains each of which is strongly separable over B, and A can be imbedded in a Galois extension of B (cf. [10]).

Moreover, by the same method as in the proof of [6, Th. 4. 3], we can prove the following

Theorem 3. *Let B be a separably closed domain. If A is an f. g.*

separable B-algebra then there exists an idempotent e of A such that $A=Ae\oplus t(A)$ where $t(A)$ is the B-torsion submodule of A and Ae is a direct sum of separably closed domains each of which is strongly separable over B.

In the rest of this note, we shall make some remarks on separable algebras over integrally closed domains. It is clear that every integrally closed domain is a separably closed domain. We now prove

Theorem 4. *B is an integrally closed domain if and only if Ω is a domain and is integrally closed.*

Proof. Let B be an integrally closed domain. Then Ω is a domain by Th. 2. Let a/b be an element of the quotient field of Ω which is integral over Ω where $a, b\in\Omega$. Then $a/b\in Q[a, b]\cap\bar{B}\subset Q[\Omega]\cap\bar{B}$. Since $J(\mathfrak{G}(Q[\Omega]\cap\bar{B}/B))=Q\cap\bar{B}=B$, it follows from Prop. 1 that $a/b\in J(\mathfrak{G}(Q[\Omega]\cap\bar{B}/B[a, b]))\subset\Omega$. Hence Ω is integrally closed. Conversely, if Ω is a domain and is integrally closed then we have $\Omega=Q[\Omega]\cap\bar{B}$ and $B=J(\mathfrak{G}(Q[\Omega]/Q)|\Omega)=J(\mathfrak{G}(Q[\Omega]/Q)|Q[\Omega]\cap\bar{B})=Q\cap\bar{B}$; hence B is integrally closed.

The following corollary contains the results of [6, Coro. 4. 2] and [7, Coro. 5. 2].

Corollary 6. *Let B be an integrally closed domain. Then every locally f. g. separable extension of B without proper idempotents is an integrally closed domain which is a subring of Ω. In particular, every f. g. separable extension of B without proper idempotents is an integrally closed domain which is projective over B.*

Proof. If R is a locally f. g. separable extension of B without proper idempotents then Ω is the separable closure of R. Hence by Th. 4, R is an integrally closed domain.

Remark 4. Let B be an integrally closed domain, and R an intermediate ring of Ω/B. Then, by Coro. 4 and Coro. 6, R is integrally closed if and only if R is locally f. g. separable over B; this is equivalent to that R is separably closed. Hence by Coro. 5, there exists a $1-1$ dual correspondence between integrally closed, intermediate rings of Ω/B and subgroups of $\mathfrak{G}(\Omega/B)$ which are closed in the finite topology, in the usual sense of Galois theory. However, in general, the subring Γ of $\bar{B}$ consisting of elements which are separable over Q is an integrally closed domain containing Ω, and there exists a $1-1$ dual correspondence between integrally closed, intermediate rings Γ/B and subgroups of $\mathfrak{G}(\Gamma/B)$ which are closed in the finite topology, in the usual sense of Galois theory.

Remark 5. Let B be an integrally closed domain. If A is an f. g. separable B-algebra then there exists an idempotent e of A such that $A = Ae \oplus t(A)$ where $t(A)$ is the B-torsion submodule of A and Ae is a direct sum of integrally closed domains each of which is strongly separable over B. This contains the result of [6, Th. 4. 3].

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