ON q-REGULAR INJECTIVE STRUCTURES IN mod-R

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Let R be a ring with identity and F an "*idempotent topologizing*" of right ideals. Then Bourbaki [1] (see also, Gabriel [2]) has shown that there exists a left exact functor $F: \mathscr{M}od\text{-}R \ni M \longmapsto F(M) \in \mathscr{M}od\text{-}R$. $F(M)$ is called a *generalized localization* of M by F, which is nothing but a *rational completion* in the sense of [3]. On the other hand, given a torsion radical, Maranda [3] has defined a quotient ring and module by a special regular injective structure determined by the torsion radical.

In this note, we shall define a *q-regular injective structure* and show that in a commutative ring every localization determines uniquely a q-regular injective structure with functorial description.

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Let r be a radical functor of the category of all right R-modules $\mathscr{M}od\text{-}R$. We shall define the following classes of right R-modules:

$\mathscr{S}_r = \{A:$ right ideal of $R \mid r(R/A) = R/A\}$.

$\widetilde{\mathfrak{Q}}_r = \{M \in \mathscr{M}od\text{-}R \mid M$ is injectve with respect to the canonical inclusion of right ideals in $\mathscr{S}_r$ in $R\}$,

$\mathfrak{Q}_r = \{M \in \mathscr{M}od\text{-}R \mid M \in \widetilde{Q}_r$ and $r(M) = 0\}$,

$\mathfrak{A}_r = \{M \in \mathscr{M}od\text{-}R \mid r(M) = 0\}$.

Definition 1. Let $\mathfrak{V}(\mathfrak{S}, \mathfrak{Q})$ be a regular injective structure and r a radical functor. We shall say that $\mathfrak{V}(\mathfrak{S}, \mathfrak{Q})$ is *defined* by r if $\mathfrak{Q}$ is a subclass of $\mathfrak{A}_r$.

For a regular injective structure $\mathfrak{V}(\mathfrak{S}, \mathfrak{Q})$ of $\mathscr{M}od\text{-}R$, let $\mathrm{Cat}(\mathfrak{Q})$ be the full subcategory of $\mathscr{M}od\text{-}R$ with object class $\mathfrak{Q}$. Then we have a functor

$$G: \mathscr{M}od\text{-}R \longrightarrow \mathrm{Cat}(\mathfrak{Q})$$

and a natural transformation

$$\kappa: I_{\mathscr{M}od\text{-}R} \longrightarrow G$$

such that the R-homomorphism $\kappa_M: M \longrightarrow G(M)$ is in $\mathfrak{S}$ for all $M \in \mathscr{M}od\text{-}R$.

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(G, κ) will be called a *functorial description* of $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ following [3; p. 107].

In [3], Maranda proved the following:

Theorem ([3; Th. 6.2]). If r is a torsion radical of $\mathcal{Mod}$-R, then $\mathfrak{Q}_r$ is the class of injectives of the coarsest regular injective structure $\mathfrak{B}_r(\mathfrak{S}_r, \mathfrak{Q}_r)$ of $\mathcal{Mod}$-R defined by the radical r.

From this fact, if $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ is a regular injective structure defined by a torsion radical r, $\mathfrak{Q}$ contains $\mathfrak{Q}_r$.

Definition 2. A regular injective structure $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ is called a *q-regular injective strncture* defined by a torsion radical t if $\tilde{\mathfrak{Q}}_t \supseteq \mathfrak{Q} \supseteq \mathfrak{Q}_t$ and has a functorial description (G_t, κ). Two q-regular injective structures $\mathfrak{B}_t(\mathfrak{S}_t, \mathfrak{Q}_t)$ and $\mathfrak{B}'_t(\mathfrak{S}'_t, \mathfrak{Q}'_t)$ defined by t are *equivalent* if there is a natural equivalence $\eta : G_t \longrightarrow G'_t$ such that $\eta_M \kappa_M = \kappa'_M$ for all $M \in \mathcal{Mod}$-R, where (G_t, κ) and (G'_t, κ') are functorial descriptions of $\mathfrak{B}_t(\mathfrak{S}_t, \mathfrak{Q}_t)$ and $\mathfrak{B}'_t(\mathfrak{S}'_t, \mathfrak{Q}'_t)$ respectively.

Throughout the following we consider a q-regular injective structure $\mathfrak{B}(\mathfrak{S}, \mathfrak{Q})$ with functorial description (G_t, κ) defined by a torsion radical t. Here $G_t(R)$ has a ring structure and $M (M \in \mathfrak{Q})$ has a right $G_t(R)$-module structure [3; p. 113—114].

Now let t be a torsion radical. Then $\mathcal{S}_t$ satisfies the condition G1-G4 in [3; p. 119]. Therefore $\mathcal{S}_t$ is an idempotent topologizing. For any right R-module M, the generalized localization $F_t(M)$ of M by $\mathcal{S}_t$ is constructed as follows: Let A, B be in $\mathcal{S}_t$ such that $A \subseteq B$. For any $M \in \mathcal{Mod}$-R, the canonical injection $j : A \longrightarrow B$ defines a homomorphism of commutative groups

$$u_{A,B} = \mathrm{Hom}_R(j, 1_M) : \mathrm{Hom}_R(B, M) \longrightarrow \mathrm{Hom}_R(A, M).$$

Since $\mathcal{S}_t$ is an ordered set with respect to the inclusion $\supseteq$, the $u_{A,B}$ define an inductive system of commutative groups, hence we define

$$M_{(F)} = \varinjlim \mathrm{Hom}_R(A, M).$$

Let $h : M \longrightarrow M_{(F)}$ be the canonical homomorphism. We consider the canonical homomorphism

$$j'_M : M \longrightarrow M/h^{-1}(0) \longrightarrow (M/h^{-1}(0))_{(F)},$$

and set $F_t(M) = (M/h^{-1}(0))_{(F)}$. Then $j^t : I_{\mathcal{Mod}\text{-}R} \longrightarrow F_t$ is a natural transformation. Note that $h^{-1}(0) = \{m \in M \mid (0 : m)_r \in \mathcal{S}_t\}$, where $(0 : m)_r$

is the right annihilator of m. Hence $h^{-1}(0)$ is the torsion submodule $t(M)$ of M with respect to t.

Lemma. *Let M be a right R-module. Then $F_t(M)$ is contained in Q_t.*

Proof. For any $A\in\mathcal{S}_t$, let $i_A : A\longrightarrow R$ be the canonical injection and $f: A\longrightarrow F_t(M)$ be any homomorphism in $\mathcal{M}od$-R. Then we have $F_t(A)=F_t(R)$ [1; p. 160] and $F_t(M)=F_t(F_t(M))$ [1; p. 161]. Since $j_R^t i_A = j_A^t$, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{i_A} & R & \xrightarrow{j_R^t} & F_t(R)=F_t(A) \\ & & {\scriptstyle f}\big\downarrow & & & \swarrow {\scriptstyle F_t(f)} & \\ & & F_t(M)=F_t(F_t(M)) & & & & \end{array}$$

Hence we have $F_t(M)\in\tilde{\mathfrak{Q}}_t$, and so $F_t(M)$ is in $\mathfrak{Q}_t$ [1; p. 160].

Proposition 1. *There is a natural transformation*

$$\chi: G_t\longrightarrow F_t.$$

Proof. Let M be any R-module. By the above lemma, there is a unique homomorphism χ_M: $G_t(M)\longrightarrow F_t(M)$ in $\mathcal{M}od$-R such that $\chi_M\kappa_M = j_M^t$. We must show that χ is a natural transformation. For any $f: M\longrightarrow N$ in $\mathcal{M}od$-R, we have

$$F_t(f)\chi_M\kappa_M = F_t(f)j_M^t = j_N^t f, \quad \chi_N G_t(f)\kappa_M = \chi_N\kappa_N f = j_N^t f.$$

Then by the uniqueness, we obtain $F_t(f)\chi_M = \chi_N G_t(f)$. This shows that χ is a natural transformation.

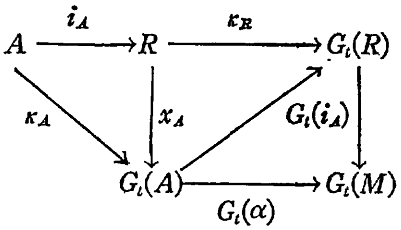
Proposition 2. *Let i_A: $A\longrightarrow R$ be the canonical inclusion, where $A\in\mathcal{S}_t$. If $G_t(i_A)$ is an epimorphism for all $A\in\mathcal{S}_t$, then there exists a natural transformation*

$$\theta_M : M_{(F)}\longrightarrow G_t(M),$$

where $M\in\mathcal{M}od$-R.

Proof. Let $A\in\mathcal{S}_t$, $M\in\mathcal{M}od$-R, and α: $A\longrightarrow M$. Then there is a homomorphism $x_A: R\longrightarrow G_t(A)$ such that $\kappa_A = x_A i_A$.

Now let us consider the following diagram:



Then there exists a unique homomorphism

$$\varphi_\alpha :\ G_t(R) \longrightarrow G_t(M)$$

such that $G_t(\alpha)x_A = \varphi_\alpha \kappa_R$. Thus we have

$$G_t(\alpha)\kappa_A = G_t(\alpha)x_A i_A = \varphi_\alpha \kappa_R i_A = \varphi_\alpha G_t(i_A)\kappa_A,$$

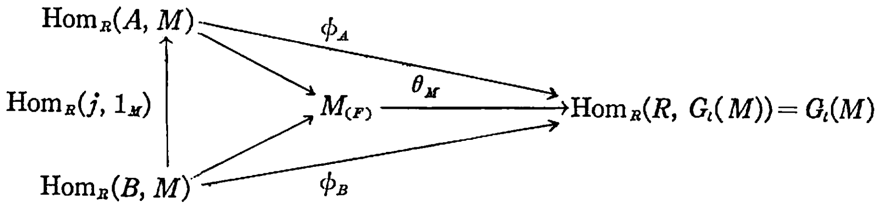
and so $G_t(\alpha) = \varphi_\alpha G_t(i_A)$. Since $G_t(i_A)$ is an epimorphism, φ_α is uniquely determined by α. Hence we have a homomorphism

$$\phi_A :\ \mathrm{Hom}_R(A, M) \ni \alpha \longmapsto \varphi_\alpha \kappa_R \in \mathrm{Hom}_R(R, G_t(M)).$$

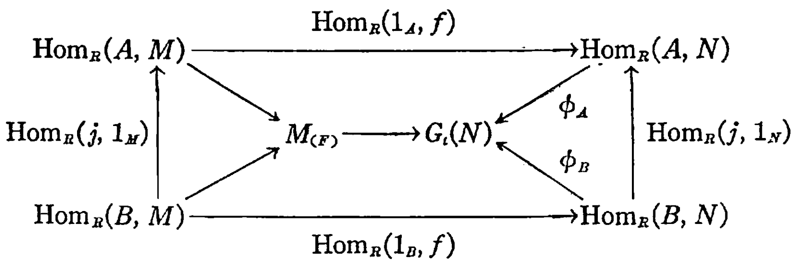
On the other hand, for the canonical inclusion $j : A \longrightarrow B (A, B \in \mathscr{S}_t)$, and for any $\beta : B \longrightarrow M$, we have

$$\begin{aligned} G_t(\beta)G_t(j)\kappa_A &= G_t(\beta j)\kappa_A = \varphi_{\beta j} G_t(i_A)\kappa_A \\ &= G_t(\beta)\kappa_B j = \varphi_\beta G_t(i_A)\kappa_A, \end{aligned}$$

whence it follows $\varphi_{\beta j} = \varphi_\beta$. From this fact, we can obtain a unique homomorphism $\theta_M : M_{(F)} \longrightarrow G_t(M)$ such that the diagram



is commutative. Similary, for any $f : M \longrightarrow N$ in $\mathcal{M}od$-R, we have a unique homomorphism $M_{(F)} \longrightarrow G_t(N)$ such that the diagram



is commutative. Therefore θ is a natural transformation.

Corollary 1. *Let M be a right R-module. If $t(M)=0$, or if $M_{(F)} = F_t(M)$, then*

$$\theta_M : M_{(F)} \longrightarrow G_t(M)$$

is an isomorphism.

Proof. If $t(M)=0$, then $M_{(F)}=F_t(M)$. Therefore, if we identify M and $\mathrm{Hom}_R(R, M)$, then we have $\theta_M j'_M = \kappa_M$. By the uniqueness and Proposition 1, it is easy to see that $\theta_M \chi_M = 1_{G_t(M)}$ and $\chi_M \theta_M = 1_{M_{(F)}}$.

Corollary 2. *Let $i_A: A \longrightarrow R$ be the canonical inclusion $(A \in \mathscr{S}_t)$. Suppose $G_t(i_A)$ is an epimorphism for all $A \in \mathscr{S}_t$.*

(1) *If $M_{(F)}=F_t(M)$ for all M, then χ_R: $G_t(R) \longrightarrow F_t(R)$ is a unique ring isomorphism such that $\chi_R \kappa_R = j'_R$, and hence the functor G_t and F_t are naturally equivalent.*

(2) *If $G_t(M) = G_t(M/t(M))$ for all M, then $\chi : G_t \longrightarrow F_t$ is a natural equivalence.*

Proof. (1): By [3; Th. 5. 3], it is clear that χ_R is a unique ring isomorphism such that $\chi_R \kappa_R = j'_R$. Therefore $G_t(M)$ is a right $F_t(R)$-module for all $M \in \mathscr{Mod}$-R. The latter statement is clear by Proposition 2 and Corollary 1.

(2): Since $t(\overline{M})=0$, where $\overline{M}=M/t(M)$, we have an isomorphism $G_t(\overline{M}) \cong F_t(\overline{M})$ by Corollary 1, and $G_t(M)=G_t(\overline{M})$ and $F_t(M)=F_t(\overline{M})$. Hence χ_M is an isomorphism.

Theorem. *Let M be any module in $\mathscr{Mod}$-R. Suppose $G_t(i_A)$ is an epimorphism for all $A \in \mathscr{S}_t$. If $M_{(F)}=F_t(M)$, or if $G_t(M/t(M))$, then $\mathfrak{B}_t(\mathfrak{S}_t, \mathfrak{Q}_t)$ is the only q-regular injective structure defined by t up to equivalence. In particular, $\{G_t(M) \mid M \in \mathscr{Mod}\text{-}R\} = \mathfrak{Q}_t = \{F_t(M) \mid M \in \mathscr{Mod}\text{-}R\}$.*

Finally, we consider the commutative case:

R: a commutative ring with identity,

S: a multiplicatively closed subset of R,

$S(M) = \{m \in M \mid ms=0 \text{ for some } s \in S\}$, for any $M \in \mathscr{Mod}$-R.

Then $S(*)$ is a torsion radical of $\mathscr{Mod}$-R. Therefore, we can define $\mathscr{S}_S$, $\tilde{\mathfrak{Q}}_S$, $\mathfrak{Q}_S$, $\mathfrak{A}_S$. In this case, $F_t(M)$ is the localization $S^{-1}M$ and $F_t(M)=M_{(F)}$

[1 ; p. 162] and $j_M^S = i_M^S$; $M \ni m \longmapsto m/s \in S^{-1}M$ for all $M \in \mathcal{Mod}\text{-}R$. Therefore, in a commutative ring, q-regular injective structure cited above is characterized by the functorial description (G_S, κ).

REFERENCES

[1] N. BOURBAKI : Algèbre commutative, chap. I-II. Hermann, 1962.
[2] P. GABRIEL : Des catégories abéliennes. Bull. Scc. Math. 90 (1962), 323—448.
[3] J.-M. MARANDA : Injective structures. Trans. Amer. Math. Scc. 110 (1964) 98—135.

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