UNITS IN COMMUTATIVE INTEGRAL GROUP RINGS*

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1. Introduction. Let us denote by $R(G)$ the group ring of a group G with coefficients from the ring R. Let $U(Z(G))$ be the group of units of the integral group ring $Z(G)$. It is well known (see [1], [3] and [4]) that if G is finite abelian

$$U(Z(G)) = \pm G \times F \quad \text{where}$$

F is a free group of rank $\frac{1}{2}((G:1)+1+n_2-2C)$ and n_2 is the number of elements of G of order 2 and c is the number of cyclic subgroups of G. In this note we compute $U(Z(G))$ for G arbitrary abelian and prove that

$$U(Z(G)) = \{\alpha g \mid g \in G, \alpha \in U(Z(H)) \text{ where } H \text{ is a finite subgroup of } G\}.$$

It is clear that when G is finite abelian any automorphism θ of $Z(G)$ is induced from a group automopphism i.e. $\theta(g) = \pm g_1, g_1 \in G$ for $g \in G$. We compute the group of automorphisms of $Z(G)$ when G is finitely generated abelian.

2. Units of $Z(G)$.

Lemma 1. *If I is an integral domain and G a torsion free abelian group then the unit group of $I(G)$ is $U(I) \cdot G$.*

Proof. See [5].

Lemma 2. *Suppose R is a commutative ring with 0 and 1 as its only idempotents. Suppose $X = \langle x \rangle$ is an infinite cyclic group. Then the unit group of $R(X)$ is $U(R) \cdot X$ if R contains no nonzero nilpotent elements.*

Proof. Suppose $\gamma, \mu \in R(X)$ such that $\gamma\mu = 1$. We can take $\gamma = \sum_0^s a_i x^i$ and $\mu = \sum_{-t}^r b_j x^j$, $a_s \neq 0$. We first claim that $\mu = \sum_{-s}^0 b_j x^j$, $b_{-s} \neq 0$. Let P be a prime ideal of R which does not contain a_s. Then looking at $\gamma\mu = 1$ in $R/P(X)$ we conclude by Lemma 1 that $b_{-s} \neq 0$ and similarly if $b_{-t} \neq 0$ then $a_t \neq 0$. Thus $t = s$. By the same argument if $b_j \neq 0$ for some $j > 0$

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then $a_{-j}\neq 0$. We conclude that $\gamma=\sum_0^s a_i x^i$ and $\mu=\sum_0^s b_{-j}x^{-j}$.

Next we assert that $a_i a_j=0$ and $a_i b_{-j}=0$ for $i\neq j$. Suppose $a_i a_j\neq 0$ then choose a prime ideal P such that $a_i a_j\notin P$. Then $\bar{a}_i\neq 0$ and $\bar{a}_j\neq 0$ in R/P and $\bar{\gamma}\bar{\mu}=1$ in the group ring $R/P(X)$. This is a contradiction to Lemma 1. Thus $a_i a_j=0$ and similarly $a_i b_{-j}=0$ and $b_{-i}b_{-j}=0$ for $i\neq j$.

Now we have the situation

$$(\sum_0^s a_i x^i)(\sum_0^s b_{-j}x^{-j})=1$$

$$(*) \qquad a_i b_{-j}=0=a_i a_j=b_{-i}b_{-j} \text{ for } i\neq j.$$

We can suppose $a_s b_{-s}\neq 0$. Then

$$a_0 b_0+a_1 b_{-1}+\cdots+a_s b_{-s}=1$$

Multiplying by a_s we have

$$a_s^2 b_{-s}=a_s \text{ and } (a_s b_{-s})^2=a_s b_{-s}.$$

Thus $a_s b_{-s}=1$ and from (*) $a_i=0$ for $i\neq s$. Hence

$$\gamma=a_s x^s \text{ and } \mu=b_{-s}x^{-s}. \quad \text{Q. E. D.}$$

We remind the reader that a group G is said to be residually finite if $\bigcap_N N=1$ where N runs over the normal subgroups of G with $(G:N)$ finite.

Lemma 3. *Suppose G is a residually finite group and $Z(G)$ its integral group ring. Then*

$$e\in Z(G),\ e^2=e \Rightarrow e=0 \text{ or } 1.$$

Proof. The corresponding result for G finite is well known (see [2]). Let $e=\sum_1^n \alpha_i g_i$, $\alpha_i\in Z$. Since G is residually finite one can choose a normal subgroup N of G such that $(G:N)<\infty$ and the cosets $g_i N$ $1\leq i\leq n$ are all distinct. Consider the equation $e^2=e$ in $Z(G/N)$. It follows then, say $g_1\in N$, $\alpha_1=1$ or 0 and $\alpha_i=0$ for $i>1$. Hence $e=0$ or 1.

Lemma 4. *Suppose G is an arbitrary abelian group. Then*

$$e\in Z(G),\ e^2=e \Rightarrow e=0 \text{ or } 1.$$

Proof. This lemma follows from the last as G can be taken to be finitely generated and hence residually finite.

Theorem 1. *Suppose G is an arbitrary abelian group. Then the unit group of $Z(G)$ is given by*

$$U(Z(G))=\{ug \mid g\in G \text{ and } u \text{ is a unit of } Z(H) \text{ for some finite subgroup } H \text{ of } G\}.$$

Proof. Let $\gamma\in U(Z(G))$. we can suppose G is finitely generated and

$$G=T\times\langle x_1\rangle\times\cdots\times\langle x_s\rangle,\ |T|<\infty.$$

We use induction on s. Since $Z(T\times\langle x_1\rangle\times\cdots\times\langle x_{s-1}\rangle)$ has no nontrivial idempotent or nilpotent elements, we can apply Lemma 2 to conclude that $\gamma=\gamma_1x_s^i$ where γ_1 is a unit of $Z(G_1)$, $G_1=T\times\langle x_1\rangle\times\cdots\times\langle x_{s-1}\rangle$. Now by induction $\gamma_1=ug_1$ where $g_1\in G$ and u is a unit of $Z(H)$, $|H|<\infty$. Thus $\gamma=ug_1x_s^i=ug$.

Corollary. *If G is a finitely generated abelian group, say, $G=T\times F$ where T is finite and F is free. Then the unit group of $Z(G)$ is given by $U_1\times F$ where U_1 is the unit group of $Z(T)$.*

3. Automorphisms of $Z(G)$. Since for G finite abelian, the only units of finite order in $Z(G)$ are $\pm g$ for $g\in G$ (see [4]) it follows that all automorphisms of $Z(G)$ are induced from (group) automorphisms of G. We have also proved (see [5]) that all automorphisms of $Z(G)$ when G is torsion or torsion free abelian, are induced from automorphisms of G. But this need not be the case when G is mixed. For example, let $G=\langle g\rangle\times\langle x\rangle$, $g^5=1$ and $o(x)=\infty$. Then U the unit group of $Z(G)$ is given by $\pm G\times\langle u\rangle$, $o(u)=\infty$. This is because the free part in the unit group of $Z(\langle g\rangle)$ has rank one (see [1] and [3]). Define $\theta: Z(G)\to Z(G)$ by

$$\theta(\sum a_{ij}g^ix^j)=\sum a_{ij}g^iu^jx^j.$$

It is easy to see that θ is a homorphism. This is a one to one map as $0=\sum a_{ij}g^iu^jx^j \Rightarrow \sum_i a_{ij}g^iu^j=0 \Rightarrow \sum a_{ij}g^i=0 \Rightarrow a_{ij}=0.$

Let G be a finitely generated abelian group. Then $G=T\times F$, where T is finite and F is free. The unit group of $Z(G)$ is given by

$$U(Z(G))=T\times U_1\times F, \text{ where}$$
$$U(Z(T))=T\times U_1.$$

Suppose

A = Automorphism group of $Z(G)$

A_1 = Group of those automorphisms of $Z(G)$ which are induced by automorphisms of G(i. e. $\theta(g)=\pm g_1$; $g, g_1 \in G$)

A_2 = Group of those automorphisms of $T\times U_1\times F$ which keep $T\times U_1$ element wise fixed.

Theorem 2. $A=A_1\times A_2$.

Proof. Given any element θ of A, define

$$\lambda(g)=\begin{cases}\theta(g) & \text{for } g\in T\\ g & \text{for } g\in F.\end{cases}$$

Extend λ to an automorphism of $Z(G)$. Then $\lambda\in A_1$ and $\lambda^{-1}\theta$ keeps $T\times U_1$ element wise fixed and can be considered as an element of A_2.

Conversely, given $\lambda\in A_1$, $\mu\in A_2$, define

$$\theta(\textstyle\sum_i u_i f_i)=\lambda(\sum_i u_i\mu(f_i)),$$

where $f_i\in F$ are distinct and $u_i\in Z(T)$. θ is clearly a homorphism. Suppose $\theta(\sum_i u_i f_i)=0$. Then

$$0=\theta(\textstyle\sum u_i f_i)=\lambda(\sum u_i\mu(f_i)).$$

Therefore, $0=\sum u_i\mu(f_i)=\sum u_i v_i f_i'$, $v_i\in T\times U_1$, $f_i'\in F$.
Therefore $u_i=0$ due to distinctness of f_i'. Hence, $\sum u_i f_i=0$.

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