3-PRIMARY COMPONENTS OF STABLE HOMOTOPIES OF $K(Z_3, 1)$ AND HP^∞

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1. Introduction

Let p be an odd prime, A be the mod p Steenrod algebra, CP^∞ and HP^∞ be the infinitely dimensional complex and quaternion projective spaces, $H^*(X)$, M, N and Q be the reduced cohomology groups of the space X, CP^∞, $K(Z_p, 1)$ and HP^∞ with coefficient group Z_p. Let $M = Z_p[y]/Z_p$, $M_k = Z_p[y^{p-1}]\cdot y^k$, $0 < k \leqq p-2$, $M_0 = Z_p[y^{p-1}]/Z_p$, $\deg(y) = 2$; $N = (Z_p[\beta x] \otimes E(x))/Z_p$, where $E(x)$ is the exterior algebra with one generator x of degree 1 and β is the Bochstein operator, N_k be the left A-submodule of N and the Z_p-module generated by $x(\beta x)^{k+i(p-1)}$, $(\beta x)^{k+1+i(p-1)}$, $i \geqq 0$, $(p-2 \geqq k \geqq 0)$. $Q = Z_p[z]/Z_p$, $\deg(z) = 4$.

The main purpose in this paper is to determine 3-primary components of stable homotopies of $K(Z_3, 1)$ and HP^∞ by the Adams spectral sequence. Liulevicius [5] determined $\pi_i^S(HP^\infty; p)$, $i \leqq 6p-2$.

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We propose:

Conjecture. Let p be a prime (which may be 2, when α_0 denotes h_0, only in this conjecture). Then the $\mathrm{Ext}_A(Z_p, Z_p)$-submodule of $\mathrm{Ext}_A(Z_p, Z_p)$ isomorphic to $\mathrm{Ext}_A(Z_p, Z_p)/Z_p[\alpha_0]$ as a Z_p-module is isomorphic to an appropriate submodule of $\mathrm{Ext}_A(H^*(K(Z_p, 1)), Z_p)$. Let d_r be the differential in the Adams spectral sequence for S^0 and $K(Z_p, 1)$ and α' denotes the generator in $\mathrm{Ext}_A(H^*(K(Z_p, 1)), Z_p)$ corresponding to α in $\mathrm{Ext}_A(Z_p, Z_p)$. Then $d_r\alpha = \gamma$ if and only if $d_r\alpha' = \gamma'$, for all r. At last $\pi_i^S(S^0; p)/Z\{\iota\}$ is isomorphic to an appropriate submodule of $\pi_i^S(K(Z_p, 1); p)$.

This conjecture is true in case of $p=2$, $t-s \leqq 29$ (except 14 and 15) by Mahowald [7] and Aikawa [3] and in case of $p=3$, $t-s \leqq 49$ (except 34, 35, 38, 42 and 46, in whose dimensions the groups are still undetermined) by this paper. We imagine that this conjecture is true for all primes and dimensions.

2. The relationship between CP^∞ and HP^∞

Since there is the map $q: CP^\infty \longrightarrow HP^\infty$ such that $q^*(z)=y^2$, we have $P^i z^j = \binom{2j}{i} z^{j+i(p-1)/2}$. Therefore we have an isomorphism $Q \cong \sum_{i=0}^{(p-1)/2} M_{2i}$ of left A-modules. In particular, if $p=3$, then $Q=M_0$. Let C_q be the mapping cone of the map q, then we have the exact sequence:

$$\cdots \longrightarrow \mathrm{H}^n(C_q) \xrightarrow{i^*} \mathrm{H}^n(HP^\infty) \xrightarrow{q^*} \mathrm{H}^n(CP^\infty) \longrightarrow \mathrm{H}^{n+1}(C_q) \longrightarrow \cdots.$$

Since q^* is a monomorphism, i^* is trivial and in fact this sequence is a short exact sequence: $0 \longrightarrow Q \xrightarrow{q^*} M \longrightarrow M/Q \longrightarrow 0$. This is split but there is no map $HP^\infty \longrightarrow CP^\infty$ except maps homotopic to the trivial map, since $\mathrm{H}^2(HP^\infty; \mathrm{Z})=0$. This split sequence induces the following split short exact sequence:

$$0 \longrightarrow \mathrm{Ext}_A^{s,t}(M/Q, \mathrm{Z}_p) \longrightarrow \mathrm{Ext}_A^{s,t}(M, \mathrm{Z}_p) \longrightarrow \mathrm{Ext}_A^{s,t}(Q, \mathrm{Z}_p) \longrightarrow 0.$$

3. The relationship between CP^∞ and $\mathrm{K}(\mathrm{Z}_p, 1)$

Proposition 3.1. *For each $\alpha \in \mathrm{Ext}_A(N, \mathrm{Z}_p)$, there is an integer m such that $\alpha\alpha_0^m=0$.*

Proof. Let J be the left A-module freely generated 1 and β as a Z_p-module and M^n and N^n be left A-submodules of M and N generated by y^i, $i \leqq n$, and $x(\beta x)^i$, $(\beta x)^{i+1}$, $i \leqq n$, respectively. Since the connecting homomorphism in the Ext-exact sequence induced by the short exact sequence $0 \longrightarrow \mathrm{Z}_p \longrightarrow J \longrightarrow \mathrm{Z}_p \longrightarrow 0$ is right multiplication by α_0, for each $\alpha \in \mathrm{Ext}_A(J, \mathrm{Z}_p)$, there is an integer m_1 such that $\alpha\alpha_0^{m_1}=0$. It follows from the Ext-exact sequence induced by $0 \longrightarrow J \longrightarrow N^n \longrightarrow N^{n-1} \longrightarrow 0$ that for each $\alpha \in \mathrm{Ext}_A(N^n, \mathrm{Z}_p)$ there is an integer m_2 such that $\alpha\alpha_0^{m_2}=0$, if the same proposition for $\mathrm{Ext}_A(N^{n-1}, \mathrm{Z}_p)$ holds.

This proposition is the most powerful tool to determine the connecting homomorphisms in Ext-exact sequences induced by $0 \longrightarrow M_{i+1} \longrightarrow N_i \longrightarrow M_i \longrightarrow 0$ $(0 \leqq i \leqq p-2)$, where replace M_{p-1} and M_0 with M_0 and $M_0 + \mathrm{Z}_p$, respectively.

Let N_0' be the A-submodule of N_0 isomorphic to $N_0/\mathrm{Z}_p\{x\}$ as a Z_p-module and $N_k'=N_k$ $(0<k\leqq p-2)$, $N'=\sum_{k=0}^{p-2} N_k'$. Then we have the following commutative diagram with two exact rows and two exact columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & N_0' & \longrightarrow & M_0 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & N_0 & \longrightarrow & M_0+Z_p & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Z_p & = & Z_p & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Corrolary 3. 2. *Unless* $k=0$ *and* $t-s=1$, *there is no generator* $\alpha\in \mathrm{Ext}_A^{s,t}(N_k', Z_p)$ *such that* $\alpha\alpha_0^m\neq 0$ *for all* $m\geqq 0$.

Proof. By Proposition 3. 1., and the sequence $0\longrightarrow N_0'\longrightarrow N_0\longrightarrow Z_p\longrightarrow 0$.

We can determine the connecting homomorphism $\mathrm{Ext}_A^{s,t}(M_1, Z_3)\longrightarrow \mathrm{Ext}_A^{s+1,t}(M_0, Z_3)$, $t-s\leqq 50$, for the upper sequence for all generators but $\underline{\nu}_0$ by this corollary. We can know the homomorphism maps $\underline{\nu}_0$ to $\underline{e}_1\rho$ since $\mathrm{Ext}_A^{1,26}(N_0, Z_3)=0$ by constructing the enough small partial minimal resolution of N_0 over A. We denote by the same letters generators in $\mathrm{Ext}_A(N_0', Z_3)$ corresponding to generators in $\mathrm{Ext}_A(M_0, Z_3)$ and $\mathrm{Ext}_A(M_1, Z_3)$ through the homomorphisms induced by $N_0'\longrightarrow M_0$ and $M_1\longrightarrow N_0'$, respectively.

Next we want to determine $\mathrm{Ext}_A(N_0, Z_3)$. The connecting homomorphism for $0\longrightarrow N_0'\longrightarrow N_0\longrightarrow Z_p\longrightarrow 0$ maps $\underline{h}_0$ to α_0 because of Proposition 3. 1.. Therefore we easily see that this homomorphism maps $\underline{\lambda}_0$, $\underline{x}_{i,2}$ $(i=5, 8, 11)$, $\underline{x}_{8,4}$ to ρ, $x_{i,1}$, $x_{8,3}$, respectively. We denote by $\underline{e}$ the generator in $\mathrm{Ext}_A^{0,1}(N_0, Z_p)$ corresponding to $1\in \mathrm{Ext}_A^{0,1}(Z_p, Z_p)$ through the homomorphism induced by $N_0\longrightarrow Z_p$. In the other words $\underline{e}$ correspods to $x\in N_0$.

At last we determine $\mathrm{Ext}_A(N_1, Z_3)$ using the Ext-exact sequence induced by the short exact sequence $0\longrightarrow L_2\longrightarrow \bar{A}\longrightarrow N_1\longrightarrow 0$. The homomorphism induced by $L_2\longrightarrow \bar{A}$ maps α_0 to α_0 and other generators in $\mathrm{Ext}_A^{s,t}(Z_3, Z_3)$, $t-s\leqq 50$, to zero, because of Proposition 3. 1. We denote by α' the generator in $\mathrm{Ext}_A^{s,t}(N_1, Z_3)$ corresponding to $\alpha\in \mathrm{Ext}_A^{s+1,t+1}(Z_3, Z_3)$ through the homomorphism induced by $\bar{A}\longrightarrow N_1$. Then the connecting homomorphism for the above short exact sequence maps $g_{i+1,i}$, $g_{2,0}$, $b_{1,2}$, $b_{1,3}$, $b_{1,4}$ to $h_i'h_{i+1}$, $h_2'h_0+h_0'h_2$, $h_1'\lambda_1$, $h_2'\rho+\rho'h_2$, $h_2'\lambda_0+\lambda_0'h_2$, respectively. We denote by $b_{1,1}$ the generator in $\mathrm{Ext}_A(N_1, Z_3)$ corresponding to $b_{1,1}$ in $\mathrm{Ext}_A(L_2, Z_3)$ through the connecting homomorphism.

There is no generator named $\underline{\rho}$ in $\mathrm{Ext}_A(N, Z_3)$ and $\underline{\rho}\alpha$ may be decomposable for α in $\mathrm{Ext}_A(Z_3, Z_3)$. $\underline{\rho}h_3$ and $\underline{\rho}\rho$, for example, are indecomposable. But we denote them in our table as if they were decomposable.

4. The differential for $K(Z_p, 1)$

Let d_r be the differentials in the Adams spectral sequences for S^0 and $K(Z_p, 1)$. We denote by the same letter both the generator in the E_2-term and its non-trivial class in the E_r-term.

Proposition 4.1. *Let α_i be generators in* $\mathrm{Ext}_A(Z_p, Z_p)$. *$d_r(\alpha_1)=\alpha_2$ if and only if $d_r(\alpha_1')=\alpha_2'$. Furthermore in case $\alpha_1\alpha_2\neq 0$, $d_r(\alpha_1\alpha_2)=\alpha_3$ if and only if $d_r(\alpha_1'\alpha_2)=\alpha_3'$ or $d_r(\alpha_2'\alpha_1)=\alpha_3'$.*

This proposition informs us of $d_r(\alpha)$ for all r and all α in $\mathrm{Ext}_A^{s,t}(N_1, Z_3)$, $t-s\leqq 50$, but ν_r', $b_{1,1}$, $h_2'\alpha_0^i (0\leqq i\leqq 5)$, $\nu_0'\lambda_0^2$ and $h_1'h_2\alpha_0$. We notice that $d_2(h_1'h_2)=\lambda_0'h_2+h_2'\lambda_0$.

Let $q': S^1 \longrightarrow K(Z_p, 1)$ be the map corresponding to the generator in $H^1(S^1: Z_p)$ and $C_{q'}$ its mapping cone. Then by the same method in § 2 we have an exact sequence :

$$\cdots \longrightarrow \mathrm{Ext}_A^{s-1,t}(N', Z_p) \longrightarrow \mathrm{Ext}_A^{s,t-1}(Z_p, Z_p) \longrightarrow \mathrm{Ext}_A^{s,t}(N, Z_p) \longrightarrow \mathrm{Ext}_A^{s,t}(N', Z_p) \longrightarrow \cdots$$

(This sequence is not so split as in § 2. because N' is not a quotient A-module of N.) We denote by q_r' the homomorphism $E_r(S^0) \longrightarrow E_r(K(Z_p, 1))$ induced by $q_*': \mathrm{Ext}_A^{s,t-1}(Z_p, Z_p) \longrightarrow \mathrm{Ext}_A^{s,t}(N, Z_p)$.

Proposition 4.2. *If α is in* $\mathrm{Ext}_A(Z_p, Z_p)$, *then $d_r(\underline{e}\alpha)$ is in $im\,(q_r')$.*

Proof. By the naturality of the differential.

For example we use this proposition to prove that $d_r(\underline{e}h_2)=0$ for all $r\geqq 2$. We remark that it informs us of only $d_2(\underline{e}h_2)=0$ that d_r is an $E_r(K(Z_p, 1))$-homomorphism ; because $\underline{e}h_2$ is an indecomposable element in $E_3(K(Z_p, 1))$. This proposition informs us of $d_r(\underline{e}\alpha)$ For all r and all α in $\mathrm{Ext}_A^{s,t}(Z_3, Z_3)$, $t-s\leqq 49$.

In Tables 1 and 3, (Z_3) and so on denote the groups expected. In Table 4, horizontal and slanting segments mean "multiplied by α_0 and

h_j'', respectively. In Table 3, j, k and m mean the same numbers only in two neighbored groups of dimension $2n-1$ and $2n$.

Table 1.

i	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\pi_i^S(HP^\infty; 3)$	Z	0	0	0	Z	0	0	0	Z	0	0	Z_3	Z	0	0	Z_3	Z

21	22	23	24	25	27	28	29	30	32	33	34	35	36	37	38	39	40
0	0	Z_3	Z	0	(Z_9)	Z	0	0	Z	0	0	Z_3+Z_3	Z	0	Z_3	Z_{3^i} $(3\leqq i\leqq 5)$	$Z+Z_3$

41	42	44	45	46	49	50	52
0	Z_3	Z	0	0	0	0	Z

Table 2.

i	42	44
$\pi_i^S(CP^\infty; 3)$	$Z+Z_3$	$Z+Z_3+Z_3$

Table 3.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\pi_i^S(K(Z_3, 1): 3)$	Z_3	0	Z_3	Z_3	Z_9	0	Z_3	0	Z_9	Z_3	Z_9+Z_3	Z_3	Z_3+Z_3	Z_3+Z_3

15	16	17	18	19	20	21	22	23	24	25	26	27	28
Z_3	$Z_3+Z_{3^j}$ $(j=0,1,2)$	$Z_{3^{3+j}}$	0	Z_3+Z_3	Z_3+Z_3	Z_3	Z_3	Z_9+Z_3	Z_3	Z_{3^k} $(k=0,1)$	Z_{3^k}	$Z_3+Z_{3^j}$ $(j=0,1)$	Z_{3^j}

29	30	31	32	33	34	35	36	41
$Z_{3^{1+k}}+Z_3$	$Z_3+Z_{3^k}$ $(k=0,1)$	Z_3+Z_3	Z_3	Z_3	(0)	$(Z_3+Z_{3^{2+j}})$	$Z_3+Z_3+Z_{3^j}$ $(j=0,1)$	$Z_3+Z_{3^j}+Z_{3^{1+m}}$ $(j, m=0,1)$

42	43	44	45	46	47	48
$Z_3+Z_{3^j}+Z_{3^m}$	Z_3+Z_3	Z_3	$Z_3+Z_9+Z_{3^j}$ $(j=0,1)$	$Z_3+Z_3+Z_3+Z_{3^{1+j}}$	$Z_3+Z_3+Z_9$	$Z_3+Z_3+Z_3$

49	50
Z_3	$(Z_3+Z_3+Z_3)$

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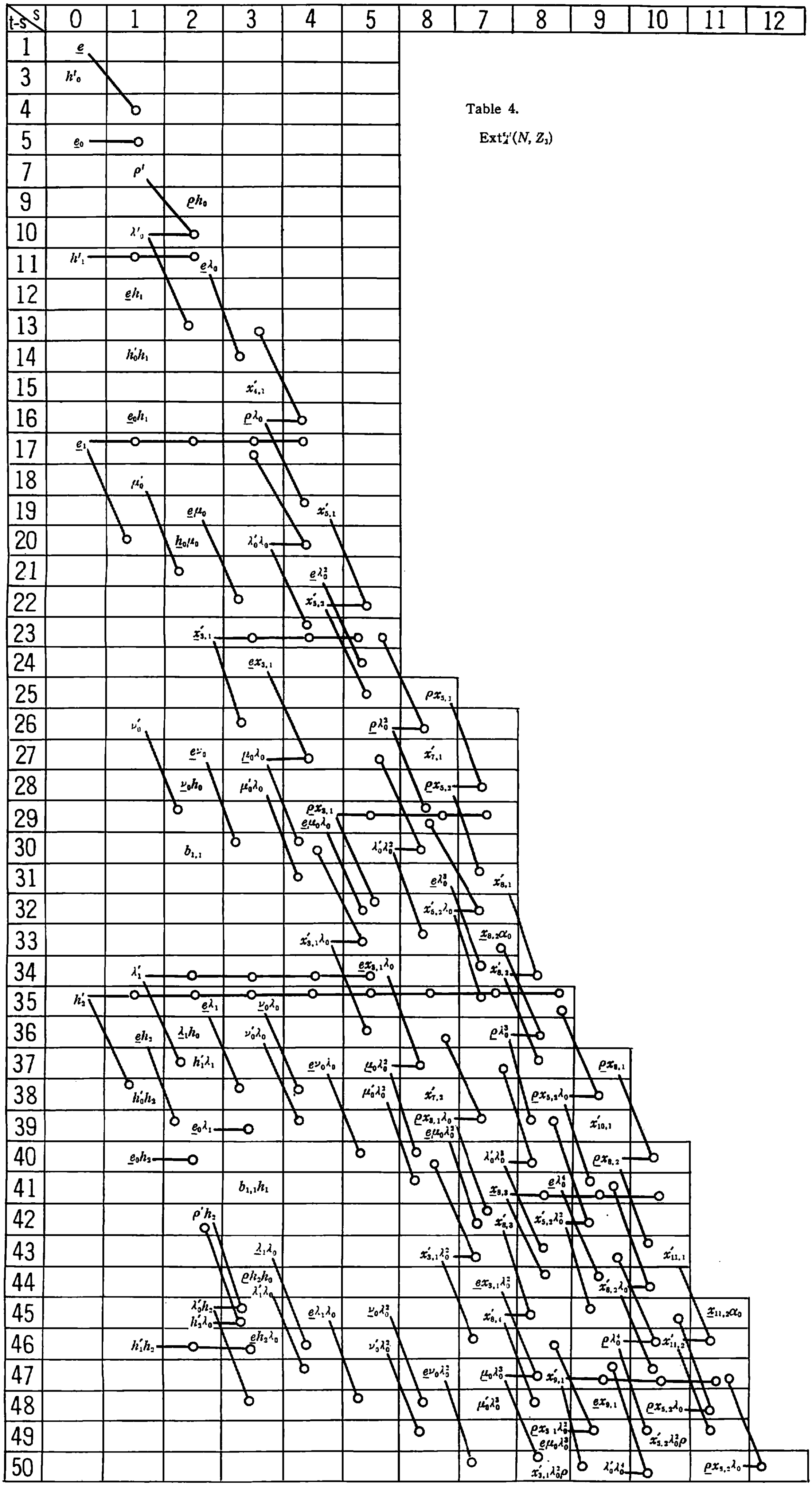

Table 4.

$\mathrm{Ext}_A^{s,t}(N, Z_3)$