SOME REMARKS ON INVARIANT SUBRINGS

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In this paper, we shall treat with subrings of a ring which are setwise invariant relative to all inner automorphisms.

Throughout our study, we use the following conventions: U will represent a ring with 1, and B a subdirectly irreducible subring of U whose unique minimal ideal T is not nilpotent. If S is a ring with 1 then $\mathfrak{R}(S)$ and $(S)_\Lambda$ will mean the Jacobson radical of S and the ring of all row-finite matrices (x_{ij}) $(x_{ij}\in S, i, j\in\Lambda)$, respectively. A unit s of S is called a *biregular element* if $1-s$ is also a unit. Following [4], the set of all biregular elements of S will be denoted by S^*, and S is defined to be *biregulary generated* (resp. *regulary generated*) if every element of S is a sum of biregular elements (resp. of units) in S. Finally, A will represent a unital subring of U[1] satisfying the following conditions:

(1) $A/\mathfrak{R}(A)\neq \mathrm{GF}(2)$, $(\mathrm{GF}(2))_2$.

(2) A is (isomorphic to) $(D)_\Lambda$ with a local ring (or a completely primary ring in the sense of [5]) D. Here, $e_{\lambda\mu}$ will represent the matrix of A with 1 in the (λ, μ)-position and 0's elsewhere. We set $E=\{\sum_{\lambda,\mu}d_{\lambda\mu}e_{\lambda\mu}\mid d_{\lambda\mu}\in D\}$ (almost all $d_{\lambda\mu}$'s are 0), and $F=\sum_\lambda Ae_{\lambda\lambda}$.

As to other notations and terminologies used in this paper, we follow the previous one [3].

1. In this section, we shall give a generalization of [3; Theorem 2].

Proposition 1. *If* $B\tilde{A}\subset B$[2] *and the right annihilator* $r_A(T)$ *of* T *in* A *is* 0 *then either* $TE\subset T$ *or* $E\subset V_U(B)$.

Proof. If a is a unit of A and $\{a, 1\}$ is not left B-free then $b_1a-b_2=0$ with some $b_1\neq 0$, $b_2\in B$, and then, noting that T is the least non-zero ideal of B and $B\tilde{A}=B$, one will easily see $Ta\subset T$. Next, if a is biregular and $\{a, 1\}$ is left B-free, then for every $b\in B$, $ab=b'a$ and

1) A unital subring of U means a subring containing the identity element of U.

2) $\tilde{A}$ represents the multiplicative group of all inner automorphisms of U induced by units of A.

$(1-a)b=b''(1-a)$ $(b', b''\in B)$ yield $b=b''=b'$, whence it follows that a is contained in $V_U(B)$. Accordingly, we have $A^*\subset A_0\cup V_U(B)$ where $A_0=\{a\in A \mid Ta\subset T\}$. Now, we shall distinguish between two cases:

(1) $\sharp\Lambda=1$: Suppose that $A\not\subset V_U(B)$ and $A\not\subset A_0$. Then there exist some $a_1\in A^*\setminus V_U(B)$ and $a_2\in A^*\setminus A_0$. By the above remark, $a_1\in A_0$, $a_2\in V_U(B)$ and then we can easily see that $\{a_2, a_1\}$ is left B-free. If a_1+a_2 is a unit of A then for every $b\in B$ $(a_1+a_2)b=b'(a_1+a_2)$ $(b'\in B)$ and $a_2b=ba_2$ yield $(b'-b)a_2+(b'-a_1ba_1^{-1})a_1=0$, which means $b'=b$. On the other hand, if a_1+a_2 is not a unit then $1-(a_1+a_2)=(-a_1)+(1-a_2)$ is a unit of A and the above argument proves again $a_1+a_2\in V_U(B)$. But this contradicts $a_1\notin V_U(B)$ and $a_2\in V_U(B)$.

(2) $\sharp\Lambda>1$: If R is an arbitrary subring of U with $R\tilde{A}\subset R$ and $Re_{ll}\subset R$ for some $l\in\Lambda$ then $RE\subset R$. In fact, for every $\lambda\neq l$ and every $d\in D$ we obtain $(1-de_{l\lambda})Re_{ll}(1-de_{l\lambda})^{-1}=R(e_{ll}+de_{l\lambda})$ and $(1+de_{\lambda l})Re_{ll}(1+de_{\lambda l})^{-1}=R(e_{ll}+de_{\lambda l})$. Combining these with $Re_{ll}\subset R$, we readily obtain $Rde_{l\lambda}$, $Rde_{\lambda l}\subset R$, whence it follows $RE\subset R$. If a_1 and a_2 are biregular elements of A such that $a_1a_2\neq a_2a_1$ and $a_1-a_2=e_{ll}$ for some $l\in\Lambda$, then by $r_A(T)=0$ and $A^*\subset A_0\cup V_U(B)$ we see that either (both a_1 and a_2 and hence) $a_1-a_2=e_{ll}$ is in A_0 or in $V_U(B)$, and hence, recalling that $T\tilde{A}=T$ and $V_U(B)\tilde{A}=V_U(B)$, the remark stated just above proves that $E\subset A_0$ or $E\subset V_U(B)$. In what follows, we shall show that we can find such biregular elements a_1 and a_2. To this end, we shall distinguish between four cases:

(i) D^* is non-empty and $\sharp\Lambda<\aleph_0$: Let d be an arbitrary element of D^*. Then, the following elements are requested ones:

$$a_1=\begin{pmatrix} 1 & 1 & & & \\ d & 0 & & & \\ & & d & & \\ & & & \ddots & \\ & & & & d \end{pmatrix}, \quad a_2=\begin{pmatrix} 0 & 1 & & & \\ d & 0 & & & \\ & & d & & \\ & & & \ddots & \\ & & & & d \end{pmatrix}.$$

(ii) D^* is non-empty and $\sharp\Lambda\geq\aleph_0$: One may regard $(D)_\Lambda$ as $((D)_2)_\Lambda$. Now, let d be an arbitrary element of D^*, and choose an arbitrary index $l\in\Lambda$. We set $a_1=(x_{\lambda\mu})$ where $x_{ll}=\begin{pmatrix}1&1\\d&0\end{pmatrix}$, $x_{\lambda\lambda}=\begin{pmatrix}d&0\\0&d\end{pmatrix}$ for $\lambda\neq l$ and $x_{\lambda\mu}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ for $\lambda\neq\mu$, and $a_2=(y_{\lambda\mu})$ where $y_{ll}=\begin{pmatrix}0&1\\d&0\end{pmatrix}$, $y_{\lambda\lambda}=\begin{pmatrix}d&0\\0&d\end{pmatrix}$ for $\lambda\neq l$ and $y_{\lambda\mu}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ for $\lambda\neq\mu$. Then, a_1 and a_2 are elements requested.

(iii) D^* is empty (i. e. $D/\mathfrak{R}(D)=\mathrm{GF}(2)$) and $2<\sharp\Lambda<\aleph_0$: In this case, we can take

$$a_1=\begin{pmatrix} 1 & & & & 1 \\ 1 & 1 & & & \\ & \cdot & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \quad \text{and} \quad a_2=\begin{pmatrix} 0 & & & & 1 \\ 1 & 1 & & & \\ & \cdot & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

(iv) D^* is empty and $\sharp\Lambda\geqq\aleph_0$: Let l be an arbitrary index in Λ. Regarding $(D)_\Lambda$ as $((D)_3)_\Lambda$ we consider $a_1=(x_{\lambda\mu})$ where $x_{ll}=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $x_{\lambda\lambda}=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ for $\lambda\neq l$ and $x_{\lambda\mu}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for $\lambda\neq\mu$, and $a_2=(y_{\lambda\mu})$ where $y_{\lambda\lambda}=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $y_{\lambda\mu}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for $\lambda\neq\mu$. Then, a_1 and a_2 are elements requested.

Corollary 1. *Let a unital subring R of U be strongly primary (or primary, semi-perfect in the sense of* [1]). *If $B\tilde{R}\subset B$ and $r_A(T)=0$ then either $TA\subset T$ or $A\subset V_U(B)$.*

The next contains evidently [3; Theorem 2].

Theorem 1. *Assume that $\mathfrak{R}(D)$ is left T-nilpotent (or right vanishing in the sense of* [6]). *If $B\tilde{A}\subset B$ and $r_A(T)=0$ then either $TA\subset T$ or $A\subset V_U(B)$.*

Proof. By Proposition 1, $TE\subset T$ or $E\subset V_U(B)$. As $\mathfrak{R}(A)=(\mathfrak{R}(D))_\Lambda$ by [6; Theorem 1], [7; Theorem] enables us to see that A is regularly generated. Especially, every element of F is a sum of units in A and $T\tilde{A}\subset T$ implies $TA=AT$. If u is an arbitrary unit of A then for every $\lambda\in\Lambda$ we can find an element $e_\lambda\in E$ such that $ue_{\lambda\lambda}=e_{\lambda\lambda}\tilde{u}\cdot u=e_{\lambda\lambda}\tilde{u}\cdot e_\lambda$, where $e_{\lambda\lambda}\tilde{u}=ue_{\lambda\lambda}u^{-1}\in F$. If $TE\subset T$ then $T\tilde{A}\subset T$ implies $Tue_{\lambda\lambda}=T\cdot e_{\lambda\lambda}\tilde{u}\cdot e_\lambda=(Te_{\lambda\lambda})\tilde{u}\cdot e_\lambda\subset Te_\lambda\subset T$. Hence, $0\neq TF=T\cdot\sum_\lambda Fe_{\lambda\lambda}\subset T$. Accordingly, we have $TFT=T$ and $TA=TFTA=TFAT=TFT=T$. On the other hand, if $E\subset V_U(B)$ then by $V_U(B)\tilde{A}\subset V_U(B)$ we have $ue_{\lambda\lambda}=e_{\lambda\lambda}\tilde{u}\cdot e_\lambda\in V_U(B)$, which implies $F=\sum_\lambda Fe_{\lambda\lambda}\subset V_U(B)$. Now, for every unit a of A, $b\in B$ and $f\in F$ there holds $f(b-aba^{-1})=fb-(fa)ba^{-1}=fb-b(fa)a^{-1}=0$. Noting that $TF\neq 0$ implies $r_B(F)=0$, we obtain $ba=ab$. Since A is regularly generated, the last means $A\subset V_U(B)$.

2. In this section, we shall give a slight improvement of [5; § 42] (cf. [2]).

Theorem 2. *Let S be a biregularly generated ring with* 1, *and R a two-sided simple unital subring of S. If $R\tilde{S}\subset R$ then $R=S$ or $R\subset V_S(S)$.*

Proof. As was noted in the proof of Proposition 1, $S^*\subset R\cup V_S(R)$. Hence, $S=R+V_S(R)=R\cdot V_S(R)=R\otimes_Z V_S(R)$, where Z is the center of R. Noting that Z is a field, one will readily see that $R=Z$ or $V_S(R)=Z$, namely, $S=V_S(R)$ or $S=R$.

The argument used in the proof of Theorem 1 suggests the following:

Theorem 3. *Let R be a two-sided simple unital subring of A. If $R\tilde{A}\subset R$ then either $R=A$ or $R\subset V_A(A)$.*

Proof. As evidently $r_A(R)=0$, Proposition 1 implies $E\subset R$ or $E\subset V_A(R)$. In case $\sharp\Lambda=1$, there is nothing to prove. Thus, in what follows, we may restrict our attention to the case $\sharp\Lambda>1$. Now, let $f=(a_{\lambda\mu})$ be an arbitrary element of F. Then there exists a finite subset I of Λ such that $\sharp I>1$ and $a_{\lambda\mu}=0$ for all $\mu\in\Lambda\backslash I$. By [7; Theorem], $f'=(a_{\lambda\mu})$ $(\lambda,\mu\in I)$ is a sum of two units in $(D)_I: f'=(a'_{\lambda\mu})+(a''_{\lambda\mu})$ $(\lambda,\mu\in I)$. We consider here the elements $u=(u_{\lambda\mu})$ and $v=(v_{\lambda\mu})$ in A which are defined as follows:

$$u_{\lambda\mu}=\begin{cases} a'_{\lambda\mu} & \text{if } \lambda,\mu\in I \\ 1 & \text{if } \lambda=\mu\in\Lambda\backslash I \\ a_{\lambda\mu} & \text{if } \lambda\in\Lambda\backslash I \text{ and } \mu\in I \\ 0 & \text{elsewhere} \end{cases}$$

and

$$v_{\lambda\mu}=\begin{cases} a''_{\lambda\mu} & \text{if } \lambda,\mu\in I \\ -1 & \text{if } \lambda=\mu\in\Lambda\backslash I \\ 0 & \text{elsewhere} \end{cases}$$

To be easily seen, u and v are units of A and $f=u+v$. If $E\subset R$ then, as was noted in the proof of Theorem 1, there holds $RF\subset R$. Accordingly, $A=RA=RFRA=RFA=R$. On the other hand, if $E\subset V_A(R)$ then in the proof of Theorem 1 we have seen that $F\subset V_A(R)$. Moreover, if $f\in F$, $a\in A$ and $b\in R$ then $f(ba-ab)=fba-(fa)b=b(fa-fa)=0$. Noting that $r_A(F)=0$, it follows $ab=ba$. We have proved therefore $R\subset V_A(A)$.

Lemma 1. *Let S be a biregularly generated ring with* 1, *and R an artinian semi-simple subring of S. If $R\tilde{S}\subset R$ then $V_R(R)=R\cap V_S(S)$.*

Proof. Let $V_R(R)=Z_1\oplus\cdots\oplus Z_t$, where Z_i is a field with the identity element e_i. Taking an arbitrary element $s\in S^*$, $se_1=e_is$ and $(1-s)e_1=e_j(1-s)$ for some i, j, whence it follows $(e_i-e_j)s=e_1-e_j$. If $e_1\neq e_i$ then $e_i\neq e_j$ and hence $e_is=e_i(e_i-e_j)s=e_i(e_1-e_j)=0$. This contradiction means $e_1\tilde{s}=e_1$. It follows therefore $Z_1\tilde{s}=Z_1\tilde{s}\cdot e_1\tilde{s}=Z_1\tilde{s}\cdot e_1$, namely, $Z_1\tilde{s}=Z_1$. Accordingly, for an arbitrary element $z\in Z_1$ there hold $sz=z's$ and $(1-s)z=z''(1-s)$ with some $z', z''\in Z_1$, and so $(z'-z'')s=z-z''$. If $z\neq z'$ then $z'\neq z''$, and hence $se_1=e_1s=(z'-z'')^{-1}(z-z'')\in Z_1$. But this implies a contradiction $z\tilde{s}=(e_1z)\tilde{s}=se_1\cdot z\cdot s^{-1}=z\cdot e_1\tilde{s}=ze_1=z$. We have seen therefore $sz=zs$, i. e. $Z_1\subset V_S(s)$. Similarly, we have $Z_i\subset V_S(s)$ $(i=1, \cdots, t)$. Hence, $V_R(R)\subset\bigcap_{s\in S^*}V_S(s)=V_S(S)$.

Theorem 4. *Let S be an artinian simple ring with* 1 *different from* $(\mathrm{GF}(2))_2$, *and $R\neq 0$ a left perfect subring of S.*[3] *If $R\tilde{S}\subset R$ then either $R=S$ or $R\subset V_S(S)$.*

Proof. Noting that all the primitive idempotents of S are mutually conjugate with respect to inner automorphisms, we can easily see that R is a unital subring of S. In case $S=\mathrm{GF}(2)$, there is nothing to prove. Thus, in below we may assume that S is biregularly generated ([4; Theorem]). As $R\tilde{S}\subset R$, we have $\mathfrak{N}(R)S=S\mathfrak{N}(R)$. Now, let s be an arbitrary element of $\mathfrak{N}(R)S$. Then $\mathfrak{l}=\{x-xs\,|\,x\in S\}$ is a left ideal of S and evidently $S=\mathfrak{l}+\mathfrak{N}(R)S$. By making use of the same argument as in [1; pp. 473—474], we can prove that $\mathfrak{l}=S$, namely, $\mathfrak{N}(R)\subset\mathfrak{N}(R)S\subset\mathfrak{N}(S)=0$. Hence, R being simple by $V_R(R)=R\cap V_S(S)$ (Lemma 1), Theorem 2 proves that either $R=S$ or $R\subset V_S(S)$.

Evidently, Theorem 4 contains [5; Theorem 42.4] and yields Corollaries 42.5 and 42.6 of [5], which are stated as follows:

Corollary 2. *Let S be an artinian semi-simple ring: $S=S_1\oplus\cdots\oplus S_t$, where each S_i is a simple ring different from* $(\mathrm{GF}(2))_2$. *If R is a right artinian subring of S and $R\tilde{S}\subset R$ then $R=S_{i_1}\oplus\cdots\oplus S_{i_k}\oplus C'$ with suitable S_{i_j}'s and a subring C' of the center of S.*

Corollary 3. *Let S be a right artinian primary ring with* 1 *such*

3) A ring R with 1 is called a left perfect ring if $R/\mathfrak{N}(R)$ is artinian and $\mathfrak{N}(R)$ is left T-nilpotent (cf. [1]).

that $S/\mathfrak{R}(S) \neq (\mathrm{GF}(2))_2$. *If* R *is an artinian semi-simple subring of* S *and* $R\tilde{S} \subset R$ *then* $\mathfrak{R}(S)$ *is contained in* $V_S(R)$ *and either* $S=R \oplus \mathfrak{R}(S)$ (*as module*) *or* $R \subset V_S(S)$.

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