ON BIREGULARLY GENERATED RINGS

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Let A be an algebra with 1 over a field Φ. A unit a of A is called a Φ-biregular element if there exists a non-zero element $\alpha\in\Phi$ such that $1\alpha-a$ is a unit of A. In his paper [1], A. Hattori proved the following which played an essential role in the characterization of invariant artinian subrings of an artinian simple ring (cf. also [2; Lemma 42.1]): Let A be a semi-primary algebra with 1 over a field Φ. If the residue class ring $\bar{A}$ or A modulo the Jacobson radical $\mathfrak{R}(A)$ is a simple ring and neither GF(2) nor $(\mathrm{GF}(2))_2$ or if $\bar{A}$ is not simple and no simple component of $\bar{A}$ is GF(2), $(\mathrm{GF}(2))_2$, GF(3) or GF(4), then A is generated by Φ-biregular elements of A as an additive group.

The purpose of this note is to present a slight generalization of the above. To this end, we shall introduce the following definitions: Let S be an arbitrary ring with 1. An element s of S is said to be *biregular* if both s and $1-s$ are units of S. The set of all biregular elements of S will be denoted by S^*, and S is defined to be *biregularly generated* if every element of S is expressible as a sum of elements of S^*.

Now, our theorem can be stated as follows:

Theorem. *Let A be a semi-primary ring with* 1, *and* $\mathrm{GF}(2)^{(n_1)}\oplus(\mathrm{GF}(2))_2^{(n_2)}\oplus\mathrm{GF}(3)^{(n_3)}\oplus\mathrm{GF}(4)^{(n_4)}\oplus A_1\oplus\cdots\oplus A_k$[1] *the representation of $A/\mathfrak{R}(A)$ as the direct sum of simple rings, where $A_i\neq\mathrm{GF}(2)$, $(\mathrm{GF}(2))_2$, $\mathrm{GF}(3)$, $\mathrm{GF}(4)$. If $n_1=n_2=0$, $n_3<2$ and $n_4<2$ then A is biregularly generated, and conversely.*

The proof of our theorem will be essentially due to the following lemmas:

Lemma 1. *Let A be an artinian simple ring with* 1. *If $A\neq\mathrm{GF}(2)$, $\mathrm{GF}(4)$, $(\mathrm{GF}(2))_2$, then every element of A is expressible as a sum of even elements of A^*.*

Lemma 2. *Let A be an artinian simple ring with* 1. *If $A\neq\mathrm{GF}(2)$, $\mathrm{GF}(3)$, $(\mathrm{GF}(2))_2$, then* 0 *is expressible as a sum of two elements of A^*.*

1) $\mathrm{GF}(2)^{(n_1)}$ means the direct sum of n_1 copies of GF(2).

Proof of Lemma 1. Let A be represented as the complete $n\times n$ matrix ring over a division ring D with matrix units e_{ij}'s. We shall distinguish here between five cases:

(1) $A=\mathrm{GF}(3)$: Evidently, $0=2+2+2+2+2+2$, $1=2+2$ and $2=2+2+2+2$.

(2) $D\neq \mathrm{GF}(2)$, $\mathrm{GF}(3)$, $\mathrm{GF}(4)$: Let $a=\begin{pmatrix} d_{11} & & ** \\ & \ddots & \\ * & & d_{nn}\end{pmatrix}$ be an arbitrary element of A. For every i there exists an element $d_i\in D$ such that $d_i\neq d_{ii}$, $d_{ii}-1, 0, 1$. Then a is the sum of biregular elements

$$\begin{pmatrix} d_{11}-d_1 & & 0 \\ & \ddots & \\ * & & d_{nn}-d_n\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} d_1 & & ** \\ & \ddots & \\ 0 & & d_n\end{pmatrix}.$$

(3) $n>1$ and $D=\mathrm{GF}(4)$: Evidently $D=\{0, 1, \alpha, \beta\}$, where $\alpha+\beta=1$, $\alpha\beta=1$, $\alpha^2=\beta$ and $\beta^2=\alpha$. For every $d\in D$ we have $de_{12}=(\alpha+de_{12})+\alpha$. Moreover, we obtain

$$\alpha e_{11}=\begin{pmatrix} \alpha & 1 & & & \\ 1 & 1 & & & \\ & & \alpha & & \\ & & & \ddots & \\ & & & & \alpha\end{pmatrix}+\begin{pmatrix} 0 & 1 & & & \\ 1 & 1 & & & \\ & & \alpha & & \\ & & & \ddots & \\ & & & & \alpha\end{pmatrix} \quad\text{and}$$

$$e_{11}=\begin{pmatrix} 0 & 1 & & & \\ \alpha & 0 & & & \\ & & \alpha & & \\ & & & \ddots & \\ & & & & \alpha\end{pmatrix}+\begin{pmatrix} 1 & 1 & & & \\ \alpha & 0 & & & \\ & & \alpha & & \\ & & & \ddots & \\ & & & & \alpha\end{pmatrix}.$$

One will easily see that these representations are decompositions into two biregular elements and every element of A is expressible as a sum of even biregular elements.

(4) $n>1$ and $D=\mathrm{GF}(3)$: Evidently $2e_{12}=(2+2e_{12})+2+2+2+2+2$

$$\text{and}\quad 2e_{11}=\begin{pmatrix} 0 & 1 & & & \\ 2 & 0 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & 2\end{pmatrix}+\begin{pmatrix} 2 & 1 & & \\ 0 & 2 & & \\ & & \ddots & \\ & & & 2\end{pmatrix}+\begin{pmatrix} 0 & 1 & & & \\ 1 & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & 2\end{pmatrix}+2+2+2.$$

Thus, taking (1) into mind, one will readily obtain our assertion.

(5) $n>2$ and $D=\mathrm{GF}(2)$: In this case, our assertion follows from the next decompositions:

$$e_{11}=\begin{pmatrix} 1 & & & & 1 \\ 1 & 1 & & & \\ & & \cdot 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix}+\begin{pmatrix} 0 & & & & 1 \\ 1 & 1 & & & \\ & & \cdot 0 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix},$$

$$e_{12}=\begin{cases} \begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & \ddots & 0 \\ & & & 1 & 1 \end{pmatrix}+\begin{pmatrix} 0 & & & & 1 \\ 1 & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & 0 \\ & & & 1 & 1 \end{pmatrix} & (n>3) \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}+\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}+\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}+\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} & (n=3). \end{cases}$$

Proof of Lemma 2. Let A be represented as the complete $n\times n$ matrix ring over a division ring D. We shall distinguish between three cases:

(1) $D\neq\mathrm{GF}(2)$, $\mathrm{GF}(3)$: There exists an element $d\in D$ different from $0, \pm 1$, namely, $d, -d\in A^*$.

(2) $n>2$ and $D=\mathrm{GF}(2)$: By the case (5) in the proof of Lemma 1, 0 is evidently a sum of two biregular elements.

(3) $n>1$ and $D=\mathrm{GF}(3)$: It suffices to prove the cases $n=2$ and $n=3$. If $n=2$ then $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ are in A^*, and if $n=3$ then $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$ are in A^*.

Proof of Theorem. Without loss of generality, from the beginning we may assume that $\mathfrak{R}(A)=0$. As we can easily see that GF(2), $(\mathrm{GF}(2))_2$, $\mathrm{GF}(3)\oplus\mathrm{GF}(3)$ and $\mathrm{GF}(4)\oplus\mathrm{GF}(4)$ are not biregularly generated, the converse part is evidently true. Accordingly, it remains to prove that $A=\mathrm{GF}(3)\oplus\mathrm{GF}(4)\oplus A_1\oplus\cdots\oplus A_k$ is biregularly generated. To see this it suffices to prove that every element in GF(3), GF(4) or A_i is expressible as a sum of elements of A^*. We shall distinguish therefore between these cases:

(1) a is in GF(3): As a is expressible as a sum of even elements of GF(3)* and 0 in GF(4) or in A_i is a sum of two elements of GF(4)* or of A_i^* by Lemmas 1 and 2, a is expressible as a sum of even elements of A^*.

(2) a is in A_i: As a is expressible as a sum of even elements of A_i^* and 0 in GF(4) or in each other A_j is a sum of two elements of GF(4)* or of A_j^* again by Lemmas 1 and 2, the present case comes to (1).

(3) a is in GF(4): In any rate, a is a sum of one or two elements of GF(4)*. Hence, we can reduce the problem to the cases (1) and (2).

As D. Zelinsky indicated in [3], every element of a closed primitive ring different from GF(2) is expressible as a sum of two units. This together with Lemmas 1 and 2 suggests the conjecture that if A is an artinian simple ring with 1 and $A \neq$ GF(2), GF(3), GF(4), $(\mathrm{GF}(2))_2$, $(\mathrm{GF}(2))_3$, then every element of A is expressible as a sum of two biregular elements. In fact, the conjecture can be proved if $A \neq$ GF(3), GF(4), $(\mathrm{GF}(2))_n$.

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