DISCRETE ANALYTIC DERIVATIVE EQUATIONS OF THE SECOND ORDER

SHIH-TONG TU

1. Introduction. This paper is concerned to discuss about the general solution of the discrete analytic derivative equations $\frac{\partial^2 F}{\partial z^2} - aK*F(z) = b(z)$ with the initial conditions $\frac{\partial F(0)}{\partial z} = C_2$ and $F(0) = C_1$. Throughout this paper, we need a few definition and some notations, such as "discrete analytic function", "region", "derivative", "line integral", "convolution", "double dot integral", "$A(R)$", "$*$"; and "L"; these are mentioned in [1].

In [2], Duffin and Duris has discussed about the general solution of discrete derivative equation of the first order with constant coefficient. If $a^4 \neq 16$, then the general solution of $\frac{\partial F(z)}{\partial z} - aF(z) = b(z)$ with $F(0) = C$, where $b(z) \in A(R)$ is $F(z) = C\, e(z, a) + \int_0^z e(z-t, a) : b(t)\delta t$ where C is an arbitrary constant, and $e(z, a) = \left(\frac{2+a}{2-a}\right)^x \left(\frac{2+ai}{2-ai}\right)^y$ is known as the discrete exponential function which is introduced ay Ferrand [3].

Afterwords, in [1], present author has developed a theory to general case, if $K(z) \in A(R)$, R contains the origin and $ah^2[K(0)+K(h)] \neq 8$ for $h = \pm 1$ or $\pm i$, then there exists a unique analytic function $F(z)$ in R, such that $\frac{\partial F(z)}{\partial z} - aK(z)*F(z)$ with $F(0) = C$, where $b(z) \in A(R)$. For the type of second order equations $\frac{\partial^2 F(z)}{\partial z^2} - aK*F(z) = b(z)$ with $\frac{\partial F(0)}{\partial z} = C_2$ and $F(0) = C_1$, we have analogous properties to the first order.

2. Discrete derivative equations of the type $\frac{\partial^2 F}{\partial z^2} - aK(z)*F(z) = 0$. In [4], Hayabara has shown the following theorem in operational sense.

Theorem 1. 1. $f \in A(R)$

$$\Rightarrow n! \int_0^z \int_0^{t_1} \cdots \int_0^{t_n} f(t_{n+1})\delta t_{n+1} \cdots \delta t_1 = \int_0^z (z-t)^{(n)} : f(t)\delta t$$

where $z^{(n+1)} = (n+1)\int_0^z t^{(n)}\delta t,\ z^{(0)} = 1.$

In [5], Duffin and Duris have solved a discrete Volterra integral equations.

Theorem 2. 1. *$f(z)$, $K(z)\in A(R)$ where R contains the origin*
2. $ah[G(0)+G(h)]\neq 4$ *for* $h=\pm 1$ *or* $\pm i$

$\Rightarrow$ *there exists a unique function* $F(z)\in A(R)$

such that $F(z)=f(z)+a\int_0^z G(z-t):F(t)\delta t.$

And the solution can be calculated by stepping formula (1).

$$(1)\quad F(z+h)=\frac{1}{4-ah[G(0)+G(h)]}\Big\{4f(z+h)+ah[G(0)+G(h)]F(z)+4a\int_0^z G(z+h-t):F(t)\delta t\Big\}$$

with $F(0)=f(0)$.

Theorem 3. 1. *$K(z)\in A(R)$ where R contains the origin*
2. $16-ah^3[K(0)+K(h)]=0$

$\Rightarrow$ (2) $\dfrac{\partial^2 F}{\partial z^2}-aK*F(z)=0$ *with* $\dfrac{\partial F(0)}{\partial z}=C_2$ *and* $F(0)=C_1$

has no solution for $z=h$ *if* $C_2=0,\ C_1\neq 0$ *or* $C_2\neq 0,\ C_1=0$.

Proof. Suppose, there exists a solution of (2) for $z=h$, with $C_2=0$, $C_1\neq 0$ or $C_2\neq 0$, $C_1=0$. Let $M(z)=K*F(z)$, from (2) we have

$$a\int_0^h M(t)\delta t=\int_0^h \frac{\partial^2 F}{\partial z^2}\delta z=\frac{\partial F(h)}{\partial z}-C_2 \qquad i.e.\ \frac{\partial F(h)}{\partial z}=\frac{ah}{2}M(h)+C_2.$$

By the definition of the derivative (see [1]), we have $\dfrac{\partial F(h)}{\partial z}=\dfrac{2}{h}[F(h)-C_1]-C_2$.

Therefore,
$$\frac{2}{h}[F(h)-C_1]-C_2=\frac{ah}{2}\int_0^h K(h-t):F(t)\delta t+C_2$$
$$=\frac{ah^2}{8}[K(0)+K(h)][F(h)+C_1]+C_2$$

i.e. $\{16-ah^3[K(0)+K(h)]\}F(h)=\{16+ah^3[K(0)+K(h)]\}C_1+16hC_2$. Hence, $16-ah^3[K(0)+K(h)]=0$ and if for $C_2=0$, $C_1\neq 0$ it contradicts to assumption. For $C_2\neq 0$, $C_1=0$ it is also a contradiction. Thus, this proves the theorem.

Theorem 4. *Let $K(z)$ be discrete analytic in R containing the origin. And if* $16-ah^3[K(0)+K(h)]\neq 0$ *for h equals to one of the values* ± 1 *or* $\pm i$. *Then there exists a unique function $F(z)$ discrete analytic in R such that*

$\frac{\partial^2 F}{\partial z^2} - aK*F(z)=0$ *with* $F(0)=C_1$ *and* $\frac{\partial F(0)}{\partial z}=C_2$. *And the solution of* (2) *can be calculated by the following stepping formula*:

(3) $$F(z+h)=\frac{1}{16-ah^3[K(0)+K(h)]}\Big\{16[C_2(z+h)+C_1]+ah^3[K(0)+K(h)]F(z)+16a\int_0^z G(z+h-t) : F(t)\delta t\Big\}$$

where $G(z)=z*K(z)$.

Proof. Suppose, (2) has a solution in R and let $K(z)*F(z)=M(z)$. Then we obtain $\frac{\partial F(z)}{\partial z}=a\int_0^z M(t)\delta t+C_2$,

and $F(z)=a\int_0^z\int_0^{t_1} M(t)\delta t\delta t_1+C_1+C_2z$.

By using Theorem 1, it becomes discrete Volterra integral equation,

such as $F(z)=a\int_0^z (z-t) : M(t)\delta t+C_2z+C_1=C_2z+C_1+aG*F(z)$……(4).

For a fixed chain $(z_0, \cdots, z_m)$ from 0 to z in R, we have

$$L\ F(z)=L\ (C_2z+C_1)+a\ L\int_0^z G(z-t) : F(t)\delta t$$

Since, $L\ (C_2z+C_1)=0$ (assume $a\neq 0$)

we can obtain the following four expressions (see [5] pp. 210—211)

$\{4-ai[G(0)+G(i)]\}L\ F(z)=0$

or $\{4+a[G(0)+G(-1)]\}L\ F(z)=0$

or $\{4+ai[G(0)+G(-i)]\}L\ F(z)=0$

or $\{4-a[G(0)+G(1)]\}L\ F(z)=0$.

But, $G(0)=0$ and $G(\mathrm{h})=\int_0^h (h-t) : K(t)\delta t=\frac{h^2}{4}[K(0)+\mathrm{K}(h)]$

Hence, above four expressions become the following forms respectively.

$\{16+ai[K(0)+K(i)]\}L\ F(z)=0$

or $\{16+a[K(0)+K(-1)]\}L\ F(z)=0$

or $\{16-ai[K(0)+K(-i)]\}L\ F(z)=0$

or $\{16-a[K(0)+K(1)]\}L\ F(z)=0$.

Thus. if $16-ah^3[K(0)+K(h)]\neq 0$ for h equal to one of the values ± 1 or $\pm i$, then $LF(z)=0$. This proves that if (2) has a solution in R, then this solution is discrete analytic in R. By theorem 2, there exists a unique solution $F(z)$ of (4) discrete analytic in R. And $F(z)$ is uniquely determined by the following stepping formula.

$$F(z+h)=\frac{1}{4-ah[G(0)+G(h)]}\Big\{4[C_2(z+h)+C_1]+ah[G(0)+G(h)]F(z)$$
$$+4a\int_0^z G(z+h-t):F(t)\partial t.\Big\}.$$

On the other hand, we can rewrite $F(z+h)$ into the following form.

$$(3)\quad F(z+h)=\frac{1}{16-ah^3[K(0)+K(h)]}\Big\{16[C_2(z+h)+C_1]+ah^3[K(0)$$
$$+K(h)]F(z)+16a\int_0^z G(z+h-t):F(t)\partial t\Big\}$$

where $G(z)=z*K(z)$.

(3) is the required stepping formula for finding the unique solution $F(z)$ of (2). Now it remains to prove that the function $F(z)$ which is obtained uniquely from (3), is exactly a solution of (2). Throughout the following proof, we use some notations. $\overline{K}(n)=K(n)+K(n-1)$ where n is a positive integer. And let $B=16-a\overline{K}(1)$, from (3) we obtain $BF(1)=16(C_1+C_2)+aC_1\overline{K}(1)$. Substituting $F(1)$ into (2), we easily see that (2) has a solution for $z=1$. Before we prove that (2) has a solution for $z=2, 3, 4, \cdots\cdots$, we need the following lemmas. The first is easy from (3).

Lemma 1.

$$(5)\qquad \overline{G}(n)=\sum_{i=1}^{n-1} i\overline{K}(n-i)+\frac{1}{4}\overline{K}(n)$$

$$(6)\qquad BF(n+1)=16[C_2(n+1)+C_1]+a\overline{K}(1)F(n)$$
$$+4a\sum_{j=2}^{n+1}\sum_{i=1}^{j-1} i\overline{K}(j-i)\overline{F}(n-j+2)+a\sum_{j=2}^{n+1}\overline{K}(j)\overline{F}(n-j+2)$$

Lemma 2. $p\geqq 4$

$$(7)\qquad E\equiv F(p-2)[12\overline{K}(1)+7\overline{K}(2)+2\overline{K}(3)]+[8\overline{G}(3)-12\overline{K}(1)-5\overline{K}(2)$$
$$-\overline{K}(3)]F(p-3)-4\overline{G}(3)F(p-4)+8[\overline{G}(p)\overline{F}(1)+\cdots+\overline{G}(4)\overline{F}(p-3)]$$
$$-4[\overline{G}(p+1)\overline{F}(1(+\cdots+\overline{G}(4)\overline{F}(p-2)]+\{\overline{\overline{\overline{K}}}(p+1)\overline{F}(1)+\cdots$$
$$+\overline{\overline{\overline{K}}}(4)\overline{F}(p-2)\}-4[\overline{G}(p-1)\overline{F}(1)+\cdots+\overline{G}(4)\overline{F}(p-4)]\equiv 0.$$

Proof. Rearranging the left-hand side into the polynomial with respect to $F(i)$, where $i=0, 1, \cdots, p-2$. We see easily that every coefficient of the term $F(i)$ equals zero. Thus, this lemma is proved.

Lemma 3. *For* $n\geqq 2$, *we have*

$$(8)\qquad 4\{F(n)-3F(n-1)+4F(n-2)-4F(n-3)+\cdots+(-1)^{n+1}4F(1)$$

$$+(-1)^n 2F(0)+(-1)^n C_2\}$$
$$=aK*F(n)+aK*F(n-1), \quad \textit{where } \frac{\partial F(0)}{\partial z}=C_2$$

Proof. It holds for $n=2$. Suppose, (8) is true for $n=p$.

(9) $$4\{F(p)-3F(p-1)+4F(p-2)-4F(p-3)+\cdots+(-1)^{p+1}4F(1)$$
$$+(-1)^p 2F(0)+(-1)^p C_2\}$$
$$=aK*F(p)+aK*F(p-1).$$

We want to claim that

(10) $$4\{F(p+1)-3F(p)+\cdots+(-1)^{p+2}4F(1)+(-1)^{p+1}2F(0)+(-1)^{p+1}C_2\}$$
$$=aK*F(p+1)+aK*F(p).$$

From (9) and (10), we get

(11) $$4\Big\{F(p+1)-2F(p)+F(p-1)\Big\}=a\int_0^{p+1}K(p+1-t):F(t)\partial t$$
$$+2a\int_0^p K(p-t):F(t)\partial t+a\int_0^{p-1}K(p-1-t):F(t)\partial t.$$

Therefore, for proving (10), it is sufficient to show (11).

Since $\int_0^p K(p-t):F(t)\partial t=\frac{1}{4}\sum_{r=1}^{p}\overline{K}(p-z_{r-1})\overline{F}(z_r)$, where $z_r=r$,

we have

$$W\equiv\text{Right-hand side of (11)}=\frac{a}{4}\Big\{\overline{\overline{\overline{K}}}(p+1)\overline{F}(1)+\cdots+\overline{\overline{\overline{K}}}(4)\overline{F}(p-2)\Big\}$$
$$+\frac{a}{4}\overline{\overline{\overline{K}}}(3)\overline{F}(p-1)+\frac{a}{4}[\overline{K}(2)+2\overline{K}(1)]\overline{F}(p)$$
$$+\frac{a}{4}\overline{K}(1)\overline{F}(p+1).$$

Let $V\equiv$ Left-hand side of (11).

Then, rewriting (11) into the form

$$4(W-V)=-BF(p+1)+16[2F(p)-F(p-1)]$$
$$+a\Big\{\sum_{i=1}^{p-2}\overline{\overline{\overline{K}}}(j+3)\overline{F}(p-j-1)+F(p-2)\overline{\overline{\overline{K}}}(3)+F(p-1)[\overline{K}(3)$$
$$+3\overline{K}(2)+3\overline{K}(1)]+F(p)[\overline{K}(2)+3\overline{K}(1)]\Big\},$$

from (6), we get

$$4(W-V)=2BF(p)-16[C_2(p+1)+C_1]+F(p-1)\{-9a\overline{K}(1)-2a\overline{K}(2)-16\}$$
$$+aF(p-2)\{-7\overline{K}(1)-2\overline{K}(2)\}-4a\{\overline{G}(p+1)\overline{F}(1)+\cdots$$

$$+\bar{G}(4)\bar{F}(p-2)\}+a[\bar{\bar{\bar{K}}}(p+1)\bar{F}(1)+\cdots+\bar{\bar{\bar{K}}}(4)\bar{F}(p-2)].$$

Again, from (6), we have

$$\begin{aligned}4(W-V)=&-BF(p-1)+16(C_2(p-1)+C_1)+F(p-2)\{17a\bar{K}(1)+8a\bar{K}(2)\\&+2a\bar{K}(3)\}+8a\{\bar{G}(p)\bar{F}(1)+\cdots+\bar{G}(4)\bar{F}(p-3)\}+8a\bar{G}(3)F(p-3)\\&-4a\{\bar{G}(p+1)\bar{F}(1)+\cdots+\bar{G}(4)\bar{F}(p-2)\}+a[\bar{\bar{\bar{K}}}(p+1)\bar{F}(1)+\cdots\\&+\bar{\bar{\bar{K}}}(4)\bar{F}(p-2)].\end{aligned}$$

Using (6) again, we obtain

$4(W-V)=aE$. By Lemma 2, we have proved this lemma.

Lemma 4. *If* $\dfrac{\partial^2 F(n-1)}{\partial z^2}-aK*F(n-1)=0$ *then* $\dfrac{\partial^2 F(n)}{\partial z^2}-aK*F(n)=0.$

Proof. By the definition of the derivative, we have

$$\begin{aligned}\frac{\partial^2 F(n)}{\partial z^2}&=2\left(\frac{\partial F(n)}{\partial z}-\frac{\partial F(n-1)}{\partial z}\right)-\frac{\partial^2 F(n-1)}{\partial z^2}\\&=4\left\{F(n)-F(n-1)-\frac{\partial F(n-1)}{\partial z}\right\}-aK*F(n-1)=\cdots\cdots\cdots\cdots\\&=4\left\{F(n)-3F(n-1)+4F(n-2)-\cdots+(-1)^{n+1}4F(1)+(-1)^n 2F(0)\right.\\&\left.\quad+(-1)^n\frac{\partial F(0)}{\partial z}\right\}-aK*F(n-1).\end{aligned}$$

From (8), we obtain $\dfrac{\partial^2 F(n)}{\partial z^2}-aK*F(n)=0.$ Thus, Lemma 4 is proved.

In conclusion, we have proved that (2) has a solution for the points on the positive x-axis. Also, we can prove that (2) has a solution for the points on the positive y-axis. By using similar process, we have that (2) has a solution $F(z)$ for the points on the real and imaginary axes. Following the remark of Duffin [6], a function $f\in A(R)$ is uniquely determined by its values on the real and imaginary axes. Therefore, Theorem 4 is proved.

3. Discrete derivative equations of the type $\dfrac{\partial^2 F(z)}{\partial z^2}-aK(z)*F(z)=b(z)$.

Theorem 5. *Let* $K(z)$ *be discrete analytic in* R *containing the origin. And if* $16-ah^3[K(0)+K(h)]\neq 0$ *for* h *equals to one of the values* ± 1 *or*

$\pm i$. *Then there exists a unique function* $F(z)$ *discrete analytic in* R, *such that*

(12) $\frac{\partial^2 F}{\partial z^2} - aK*F(z) = b(z)$ *with* $F(0) = C_1$ *and* $\frac{\partial F(0)}{\partial z} = C_2$, *where* $b(z) \in A(R)$. *And the solution of* (12) *can be calculated by the following stepping formula*:

(13)
$$F(z+h) = \frac{1}{16 - ah^3[K(0)+K(h)]}\Big\{16[C_2(z+h)+C_1+H(z+h)] + ah^3[K(0)+K(h)]F(z) + 16a\int_0^z G(z+h-t) : F(t)\partial t\Big\}$$

with $\frac{\partial^2 F(0)}{\partial z^2} = b(0)$, *where* $H(z) = z*b(z)$ *and* $G(z) = z*K(z)$.

Proof. Let $M(z) = K*F(z)$, from (12) we have

$$F(z) = \int_0^z \int_0^{t_1} [aM(t)+b(t)]\partial t \partial t_1 + C_1 + C_2 z$$
$$= \int_0^z (z-t) : [aM(t)+b(t)]\partial t + C_2 z + C_1$$

i. e.

(14) $F(z) = C_2 z + C_1 + H(z) + aG*F(z)$.

This is a discrete Volterra integral equation. Since $K(z) \in A(R)$, $C_2 z + C_1 + H(z) \in A(R)$ and $16 - ah^3[K(0)+K(h)] \neq 0$ is equivalent to $ah[G(0)+G(h)] \neq 4$, and by Theorem 2 we obtain that there exists a unique discrete analytic solution $F(z)$ of (14). And the solution can be calculated by the following stepping formula.

$$F(z+h) = \frac{1}{4 - ah[G(0)+G(h)]}\Big\{4[C_2(z+h)+C_1+H(z+h)] + ah[G(0)+G(h)]F(z) + 4a\int_0^z G(z+h-t) : F(t)\partial t\Big\}$$

On the other hand, we can rewrite $F(z+h)$ into the form (13). Thus (13) is the required stepping formula for finding the unique solution $F(z)$ of (12). With the similar proof of Theorem 4, we see that the function $F(z)$ which is obtained uniquely from (13) is exactly a soluton of (12).

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DEPARTMENT OF MATHEMATICS,
OKAYAMA UMIVERSITY

DEPARTMENT OF MATHEMATICS,
CHUNG YUAN CHRISTIAN COLLEGE OF SCIENCE AND ENGINEEING,
CHUNG LI, TAIWAN, CHINA

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