DISCRETE ANALYTIC DERIVATIVE EQUATIONS OF THE FIRST ORDER

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Introduction. The concept of a discrete analytic function was introduced by Jacqueline Ferrand [1], and many properties of discrete analytic functions were obtained by Duffin [2]. In what follows, we need a few definition concerning the discrete complex plane. The discrete complex plane is the set of all lattice points in the complex plane with integer coordinates. A region in the discrete complex plane is the union of unit squares $\{z, z+1, z+1+i, z+i\}$. A chain $z_0, z_1, \cdots, z_m$ is a set of points in the discrete complex plane such that $|z_i - z_{i-1}| = 1$. A region is said to be connected if any two points of the region can be combined by a chain in the region. A simple connected region R is a simply connected set which is the union of a finite number of unit squares. Thus the boundary of R is a simple closed curve which is composed of edges of unit squares. Throughout this paper, we assume that R is a rectangular region, *i.e.* rectangular region is a simple connected region. Consider a complex valued function f defined on a square $\{z, z+1, z+1+i, z+i\}$, f is said to be discrete analytic on that square if

$$Lf(z) \equiv f(z) + if(z+1) + i^2 f(z+1+i) + i^3 f(z+i) = 0.$$

If f is discrete analytic on every squares in R we denote it by $f \in A(R)$. If $f \in A(R)$, Ferrand [1] defines a discrete derivative of f denoted by $\frac{\partial f(z)}{\partial z}$ by the difference equation

$$\frac{\frac{\partial f(p)}{\partial z} + \frac{\partial f(q)}{\partial z}}{2} = \frac{f(p) - f(q)}{p - q} \tag{1}$$

where p and q are neighboring points, *i.e.* $|p-q| = 1$.

It is shown [1] that if the value of $\frac{\partial f}{\partial z}$ is specified at some fixed point in R, then (1) determines $\frac{\partial f}{\partial z}$ uniquely and $\frac{\partial f}{\partial z} \in A(R)$.

The line integral of f is defined as

(2) $$\int_a^b f(t)\,\delta t=\sum_{n=1}^{m}\frac{1}{2}[f(z_n)+f(z_{n-1})](z_n-z_{n-1}).$$

where $a=z_0, z_1\cdots, z_m=b$ is a chain of points in R connecting a to b.

It follows [2] that if $f\in A(R)$, this integral is path independent and

(3) $$f(b)-f(a)=\int_a^b\frac{\partial f}{\partial z}\,\delta z$$

If a and b belong to R, Duffin [2] defines a "double dot" line integral by

(4) $$\int_a^b f(t):g(t)\,\delta t=\sum_{n=1}^{m}\frac{1}{4}[f(z_n)+f(z_{n-1})][g(z_n)+g(z_{n-1})](z_n-z_{n-1})$$

where $a=z_0, \cdots, z_m=b$ is any chain of points in R connecting a to b.

It is shown in [2] that this integral is independent of path if f, $g\in A(R)$. Duffin and Duris [3] define convolution product by

$$f*g(z)=\int_0^z f(z-t):g(t)\,\delta t \qquad \text{for all } z\in R$$

In [3], it is shown that for f and $g\in A(R)$, a convolution product is independent of integration path, and is also commutative, associative, and distributive over usual pointwise addition.

Discrete derivative equations of the first order. In [4], Duffin and Duris has discussed about the general solution of discrete derivative equation of the first order with constant coefficent. If $a^4\neq 16$, then the general solution of

$$\frac{\partial F(z)}{\partial z}-aF(z)=b(z) \text{ with } F(0)=C, \text{ where } b(z)\in A(R)$$

is
$$F(z)=Ce(z,a)+\int_0^z e(z-t,a):b(t)\,\delta t$$

where C is an arbitrary constant, and $e(z,a)=\left(\frac{2+a}{2-a}\right)^x\left(\frac{2+ai}{2-ai}\right)^y$ is known as the discrete exponential function which is introduced by Ferrand [1], the solution $F(z)$ is defined and is single valued in R and $F(z)\in A(R)$.

Duffin and Duris do not devolope a theory to general case. In this paper we shall consider the general case, such as

(5) $$\frac{\partial F(z)}{\partial z}-aK(z)*F(z)=0$$

where $K(z)\in A(R)$ is given, and a is an arbitrary constant.

It is of interest, under what conditions, there exists a analytic solution of (5).

Theorem 1. *Let $K(z)$ be discrete analytic in R containing the origin. If $ah^2[K(0)+K(h)]=8$ for $h=\pm 1$ or $\pm i$, then the discrete analytic homogeneous derivative equations* (5) *with $F(0)=C$ has no solution for $z=h$, where C is a non-zero arbitrary constant.*

Proof. Suppose, there exists a solution of (5) for $z=h$. Putting (5) into integral form we have

$$F(h)=a\int_0^h K(z)*F(z)\delta z+F(0).$$

Let $K(z)*F(z)=G(z)$, then

$$F(h)=\frac{ah}{2}G(h)+C=\frac{ah}{2}\int_0^h K(h-t):F(t)\delta t+C$$
$$=\frac{ah^2}{8}[K(0)+K(\mathrm{h})][F(h)+F(0)]+C$$

i. e. $\{8-ah^2[K(0)+K(h)]\}F(h)=\{8+ah^2[K(0)+K(h)]\}C.$
Thus, if $ah^2[K(0)+K(h)]=8$ it contradicts to assumption.

Theorem 2. *Let $K(z)$ be discrete analytic in R containing the origin. If*

(5) $$\frac{\partial F(z)}{\partial z}-aK(z)*F(z)=0 \quad \textit{with} \quad F(0)=C$$

has a solution in R,
then this solution is discrete analytic in R provided a satisfies at least one of the following conditions:

(6) $$8+a[K(0)+K(i)]\neq 0$$

(7) $$8+ai[K(0)+K(-1)]\neq 0$$

(8) $$8-a[K(0)+K(-i)]\neq 0$$

(9) $$8-ai[K(0)+K(1)]\neq 0$$

Before proving this, we state the following lemma.

Lemma. *If* $\dfrac{\partial F(z)}{\partial z}-aK(z)*F(z)=0$ *with* $F(0)=C$, *then*
$$L\,F(z)=\frac{ia}{2}L[K(z)*F(z)]$$

Proof of Lemma. Let $G(z)=K*F(z)$, then we have

$$F(z)=a\int_0^z G(t)\partial t+C$$

$$F(z+1)=a\int_0^{z+1} G(t)\partial t+C=F(z)+a\int_z^{z+1} G(t)\partial t=F(z)+\frac{a}{2}[G(z+1)+G(z)].$$

$$F(z+1+i)=F(z)+a\int_z^{z+1} G(t)\partial t+a\int_{z+1}^{z+1+i} G(t)\partial t$$

$$=F(z)+\frac{a}{2}[G(z+1)+G(z)]+\frac{ai}{2}[G(z+1+i)+G(z+1)].$$

$$F(z+i)=F(z)+a\int_z^{z+i} G(t)\partial l=F(z)+\frac{ai}{2}[G(z+i)+G(z)].$$

i.e. $L\ F(z)=F(z)+iF(z+1)-F(z+1+i)-iF(z+i)$

$$=\frac{ai}{2}\ L\ G(z)=\frac{ai}{2}L[K(z)*F(z)].$$

Proof of Theorem 2. Let $G(z)=K*F(z)$, by Lemma, we have $L\ F(z)=\frac{ai}{2}L\ G(z)$.

(a) $$L\ G(z)=\int_0^z K(z-t):F(t)\partial t+i\int_0^{z+1} K(z+1-t):F(t)\partial t$$

$$-\int_0^{z+1+i} K(z+1+i-t):F(t)\partial t-i\int_0^{z+i} K(z+i-t):F(t)\partial t$$

Since $LK(z-t)=0$, we get

(b) $$0=\int_0^z[K(z-t)+iK(z+1-t)-K(z+1+i-t)-iK(z+i-t)]:F(t)\partial t.$$

From (a) and (b), we have

$$L\ G(z)=i\int_z^{z+1} K(z+1-t):F(t)\partial t-\int_z^{z+1+i} K(z+1+i-t):F(t)\partial t$$

$$-i\int_z^{z+i} K(z+i-t):F(t)\partial t=\frac{i}{4}[K(0)+K(i)]L\ F(z).$$

Substituting this result into $L\ F(z)$, we get

$$L\ F(z)=\frac{i^2a}{8}[K(0)+K(i)]LF(z)$$

i.e. $L\ F(z)\{8+a[K(0)+K(i)\}=0$.

If $8+a[K(0)+K(i)]\neq 0$, we have $L\ F(z)=0$, thus (6) is proved.

Similarly, from $L\ K(z-t)=0$, we have

(c) $0=\int_0^{z+1}\{K(z-t)+iK(z+1-t)-K(z+1+i-t)-iK(z+i-t)\}:F(t)\delta t.$

From (a) and (c), we get

$$L\ G(z)=\int_{z+1}^{z}K(z-t):F(t)\delta t-\int_{z+1}^{z+1+i}K(z+1+i-t):F(t)\delta t-i\int_{z+1}^{z+i}K(z+i-t):F(t)\delta t=-\frac{1}{4}[K(0)+K(-1)]\,L\ F(z).$$

Putting $LG(z)$ into $L\ F(z)$, we obtain $LF(z)\{8+ai[K(0)+K(-1)]\}=0$. If $8+ai[K(0)+K(-1)]\neq 0$, we have $LF(z)=0$ thus (7) is proved. By the same way, we can obtain the two expressions

$$LF(z)\{8-a[K(0)+K(-i)]\}=0$$

and $\quad LF(z)\{8-ai[K(0)+K(1)]\}=0\,;$

therefore, if $8-a[K(0)+K(-i)]\neq 0$ then $LF(z)=0$ and if $8-ai[K(0)+K(1)]\neq 0$ then $LF(z)=0$. This proves Theorem 2.

Cororally. 1. $K(z)\in A(R)$, *where* R *contains the origin*
2. $ah^2[K(0)+K(h)]\neq 8$ *for* $h=\pm 1$ or $\pm i$
3. (5) *has a solution in* R
$\Rightarrow$ *this solution* $\in A(R)$

Theorem 3. *Let $K(z)$ be discrete analytic in R containing the origin. If $ah^2[K(0)+K(h)]\neq 8$ for $h=\pm 1$ or $\pm i$.*
Then there exists a nnique analytic function $F(z)$ in R such that

$$(5)\qquad \frac{\partial F(z)}{\partial z}-aK(z)*F(z)=0 \quad with \quad F(0)=C.$$

Proof. The definition of the double dot line integral yields from (5) the stepping formula (10). Let $G(z)=K*F(z)$, we have

$$F(z+h)-F(z)=\frac{ah}{2}[G(z+h)+G(h)]$$
$$=\frac{ah^2}{8}[K(0)+K(h)][F(z+h)+F(z)]+\frac{ah}{2}\int_0^z K(z+h-t)+K(z-t):F(t)\delta t$$

i. e.

$$(10)\qquad F(z+h)=\frac{8+ah^2[K(0)+K(h)]}{8-ah^2[K(0)+K(h)]}F(z)+\frac{4ah}{8-ah^2[K(0)+K(h)]}\int_0^z[K(z+h-t)+K(z-t)]:F(t)\delta t$$

where h equals ± 1 or $\pm i$.

Since $F(0)=C$, we may calculate to any z in R by (10), that is $F(z)$ exists uniquely by successive substitution. By Theorem 2, we also know that $F(z)\in A(R)$. It remains to prove that $F(z)$ is a required solution. Let $\frac{\partial F(0)}{\partial z}=0$, we use the symbols $\overline{K}(1)=K(1)+K(0)$, $\overline{K}(2)=K(2)+K(1)$.

And then, from (10), we get $F(1)=\frac{8+a\overline{K}(1)}{8-a\overline{K}(1)}C$.

By the definition of the derivative (1), we have $\frac{\partial F(1)}{\partial z}=2[F(1)-C]$ $=\frac{4a\overline{K}(1)}{8-a\overline{K}(1)}C$. Since $aG(1)=a\int_0^1 K(1-t):F(t)\partial t=\frac{4a\overline{K}(1)}{8-a\overline{K}(1)}C$, we have $\frac{\partial F(1)}{\partial z}-aK*F(1)=0$. Thus, (5) has a solution for $z=1$. Similarly, from (10), we get

$$F(2)=\frac{8+a\overline{K}(1)}{8-a\overline{K}(1)}F(1)+\frac{a}{8-a\overline{K}(1)}[\overline{K}(2)+\overline{K}(1)][F(1)+C].$$

From (1), we have

$$\begin{aligned}\frac{\partial F(2)}{\partial z}&=2[F(2)-F(1)]-\frac{\partial F(1)}{\partial z}\\&=\frac{2a}{8-a\overline{K}(1)}\{[\overline{K}(2)+3\overline{K}(1)]F(1)+[\overline{K}(2)-\overline{K}(1)]C\}.\end{aligned}$$

Since $aG(2)=a\int_0^2 K(2-t):F(t)\partial t=\frac{2a}{8-a\overline{K}(1)}\{[\overline{K}(2)+3\overline{K}(1)]F(1)+[\overline{K}(2)$ $-\overline{K}(1)]C\}$, threfore, we have $\frac{\partial F(2)}{\partial z}-aK*F(2)=0$. Thus, (5) has a solution for $z=2$. By induction, it is easily proved that (5) has a solution for the points on the positive x-axis, also on the positive y-axis. And by using similar process, we have that (5) has a solution $F(z)$ for the points on the real and imaginary axis. Following the remarks of Duffin [2], a function $f\in A(R)$ is uniquely determined by its values on the real and imaginary axes. Therefore Theorem 3 is proved.

Theorem 4. 1. *$K(z)\in A(R)$, where R contains the origin*
2. *$ah^2[K(0)+K(h)]\neq 8$ for $h=\pm 1$ or $\pm i$*

$\Rightarrow$ *There exists a unique analytic function $F(z)$ in R such that $\frac{\partial F(z)}{\partial z}-aK(z)*F(z)=b(z)$ with $F(0)=C$, where $b(z)\in A(R)$.*

The proof of this theorem is similar to the proof of Theorem 3. The stepping formula is

$$F(z+h)=\frac{8+ah^2[K(0)+K(h)]}{8-ah^2[K(0)+K(h)]}F(z)$$
$$+\frac{4h\left\{a\int_0^z[K(z+h-t)+K(z-t)]:F(t)\partial t+[b(z+h)+b(z)]\right\}}{8-ah^2[K(0)+K(h)]}$$

with $\frac{\partial F(0)}{\partial z}=b(0)$.

Remark. Theorem 4 is a generalization of Theorem 3. 3 which is mentioned by Duffin and Duris [4]. Since, there exists a discrete analytic function $e(z)=(-1)^{x+y}\{-4x+4yi+e(0)\}$ such that $e*F(z)=F(z)$ with $F(0)=0$. (see [5]. p. 47).

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