ON PERVIN'S QUASI UNIFORMITY

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1. Introduction

In [1], Pervin introduced a quasi uniformity $\mathscr{U}(\mathfrak{T})$ determined by a topology $\mathfrak{T}$ on the set X by taking sets of the form $O\times O\cup \mathscr{C}O\times X$, $O\in\mathfrak{T}$ as subbase. He proved that the topology induced by $\mathscr{U}(\mathfrak{T})$ is in fact the original topology $\mathfrak{T}$. Thus every topological space is quasi uniformizable.

It is the purpose of this paper to explore more fully the relationships that exist between $\mathfrak{T}$ and $\mathscr{U}(\mathfrak{T})$. In § 2, the following topological properties are characterized in terms of $\mathscr{U}(\mathfrak{T})$: T_0, T_1, T_2, $\mathfrak{T}=\mathfrak{F}$ ($\mathfrak{F}$ denoting the class of all closed sets), $\mathfrak{T}$ is discrete, $\mathfrak{T}$ is indiscrete, $\mathfrak{T}$ has three or less elements, $(X, \mathfrak{T})$ is disconnected, X is finite and $\mathfrak{T}$ is discrete. In § 4, relationships between continuity and uniform continuity are determined and compactness is characterized in terms of $\mathscr{U}(\mathfrak{T})$. In § 5, we give an example to show that $\mathscr{U}(\times\mathfrak{T}_i)\neq\times\mathscr{U}(\mathfrak{T}_i)$.

2. Topological properties

Theorem 2.1. *$(X, \mathfrak{T})$ is a* (i) *T_0-space iff* $\Delta=\cap\{U\cap U^{-1}: U\in\mathscr{U}(\mathfrak{T})\}$ (ii) *T_1-space iff* $\Delta=\cap\{U: U\in\mathscr{U}(\mathfrak{T})\}$ *and* (iii) *T_2 space iff* $\Delta=\cap\{cU: U\in\mathscr{U}(\mathfrak{T})\}$.

Proof of (i). Suppose that $(X, \mathfrak{T})$ is a T_0-space and that $(x, y)\in U\cap U^{-1}$ for each $U\in\mathscr{U}(\mathfrak{T})$. We must show that $x=y$. Suppose on the contrary that $x\neq y$. Case 1. There exists an $O^*\in\mathfrak{T}$ such that $x\in O^*$ and $y\notin O^*$. Then $(x, y)\notin O^*\times O^*\cup\mathscr{C}O^*\times X=U^*$. Thus $(x, y)\notin U^*\cap U^{*-1}$. Case 2. There exists an $O^\#\in\mathfrak{T}$ such that $x\notin O^\#$ and $y\in O^\#$. Then $(y, x)\notin O^\#\times O^\#\cup\mathscr{C}O^\#\times X=U^\#$. Hence $(x, y)\notin U^\#\cap U^{\#-1}$.

Conversely, suppose that $\Delta=\cap\{U\cap U^{-1}: U\in\mathscr{U}\}$ and suppose that $x\neq y$. Then $(x, y)\notin U\cap U^{-1}$ for some $U\in\mathscr{U}(\mathfrak{T})$. Case 1. $(x, y)\notin U$. Then $(x, y)\notin O\times O\cup\mathscr{C}O\times X$ for some $O\in\mathfrak{T}$ and it follows that $x\in O$ and $y\notin O$. Case 2. $(x, y)\notin U^{-1}$. Then $(y, x)\notin U$ and case 1 may be applied.

Proof of (ii). Let $(X, \mathfrak{T})$ be a T_1-space and suppose that $x\neq y$. We will show that $(x, y)\notin U$ for some $U\in\mathscr{U}(\mathfrak{T})$. In fact, we may take $U=\mathscr{C}\{y\}\times\mathscr{C}\{y\}\cup\mathscr{C}\mathscr{C}\{y\}\times X$.

Conversely, let $x \neq y$ in X. Then $(x, y) \notin U^*$ for some $U^* \in \mathscr{U}(\mathfrak{T})$ and hence $(x, y) \notin O^* \times O^* \cup \mathscr{C}O^* \times X$ for some $O^* \in \mathfrak{T}$. Hence $x \in O^*$ and $y \notin O^*$.

Proof of (iii). If $\triangle = \bigcap \{cU : U \in \mathscr{U}(\mathfrak{T})\}$, then $\triangle$ is closed in $X \times X$ and $(X, \mathfrak{T})$ is T_2.

Conversely, suppose that $(X, \mathfrak{T})$ is a T_2-space and that $x \neq y$. There exist disjoint open sets O_x and O_y such that $x \in O_x$ and $y \in O_y$. Hence $(x, y) \notin cU$ where $U = O_x \times O_x \cup \mathscr{C}O_x \times X$.

Theorem 2.2. *Let $\mathfrak{F}$ denote the family of closed sets in $(X, \mathfrak{T})$. Then $\mathfrak{T} = \mathfrak{F}$ iff U is a neighborhood of $\triangle$ whenever $U \in \mathscr{U}(\mathfrak{T})$.*

Proof. If $\mathfrak{T} = \mathfrak{F}$, then $O \times O \cup \mathscr{C}O \times X$ is an open neighborhood of the diagonal for each $O \in \mathfrak{T}$. Thus U in $\mathscr{U}(\mathfrak{T})$ implies that U is a neighborhood of the diagonal.

Conversely, suppose that U in $\mathscr{U}(\mathfrak{T})$ implies that U is a neighborhood of the diagonal. Let $O \in \mathfrak{T}$. Then $O \times O \cup \mathscr{C}O \times X \in \mathscr{U}(\mathfrak{T})$ and hence there exists a $G \in \mathfrak{T} \times \mathfrak{T}$ such that $O \times O \cup \mathscr{C}O \times X \supseteq G \supseteq \triangle$. Then $O \times O \cup \mathscr{C}O \times \mathscr{C}O \supseteq G \cap G^{-1} \supseteq \triangle$ and $G \cap G^{-1} \in \mathfrak{T} \times \mathfrak{T}$. Let $x \in \mathscr{C}O$. Then $x \in G \cap G^{-1}[x] \subseteq (O \times O \cup \mathscr{C}O \times \mathscr{C}O)[x] = \mathscr{C}O$ and $\mathscr{C}O$ is open. It follows then that $\mathfrak{T} = \mathfrak{F}$.

Corollary 2.3. *$\mathscr{U}(\mathfrak{T})$ is a uniformity iff $\mathfrak{T} = \mathfrak{F}$.*

Proof. If $\mathscr{U}(\mathfrak{T})$ is a uniformity, then $U \in \mathscr{U}(\mathfrak{T})$ implies that U is a neighborhoood of $\triangle$ and hence by Theorem 2. 2, $\mathfrak{T} = \mathfrak{F}$.

Conversely, suppose that $\mathfrak{T} = \mathfrak{F}$. It suffices to show that $(O \times O \cup \mathscr{C}O \times X)^{-1} \in \mathscr{U}(\mathfrak{T})$ when $O \in \mathfrak{T}$. But $(O \times O \cup \mathscr{C}O \times X)^{-1} \supseteq O \times O \cup \mathscr{C}O \times \mathscr{C}O = (O \times O \cup \mathscr{C}O \times X) \cap (\mathscr{C}O \times \mathscr{C}O \cup O \times X) \in \mathscr{U}(\mathfrak{T})$.

Corollary 2.4. *The following are equivalent:* (i) $(X, \mathfrak{T})$ *is discrete* (ii) $\mathfrak{T} = \mathfrak{F}$ *and* $(X, \mathfrak{T})$ *is a T_0-space* (iii) $\mathscr{U}(\mathfrak{T})$ *is a uniformity and* $\triangle = \bigcap \{U \cap U^{-1} : U \in \mathscr{U}(\mathfrak{T})\}$.

Proof. (i) clearly implies (ii) and (ii) is equivalent to (iii) by corollary 2. 3 and (i) of theorem 2. 1. To show that (ii) implies (i), it suffices to show that $\{x\}$ is closed for each $x \in X$. But $\mathfrak{T} = \mathfrak{F}$ and $(X, \mathfrak{T})$ a T_0-space clearly implies that $(X, \mathfrak{T})$ is a T_2-space and hence a T_1-space.

Theorem 2.5. $(X, \mathfrak{T})$ *is trivial iff* $(X, \mathscr{U}(\mathfrak{T}))$ *is trivial.*

Proof. Exercise for the reader.

Lemma 2.6. *Let* $B\subseteq X$, $B\neq X$. *If* $A\times A\cup\mathscr{C}A\times X\subseteq B\times B\cup\mathscr{C}B\times X$, *then* $A\supseteq B$.

Proof. Let $b\in B$ and suppose that $b\notin A$. Take $q\notin B$. Then $(b,q)\in A\times A\cup\mathscr{C}A\times X$, but $(b,q)\notin B\times B\cup\mathscr{C}B\times X$, a contradiction.

Lemma 2.7. *Let* $\emptyset\neq B\subseteq X$ *and suppose that* $A\times A\cup\mathscr{C}A\times X\subseteq B\times B\cup\mathscr{C}B\times X$. *Then* $A\subseteq B$.

Proof. Case 1. $B=X$. Then $A\subseteq B$. Case 2. $B\neq X$. Then by Lemma 2.6, $A\supseteq B$. Now suppose that $A\not\subseteq B$. Take $a\in A$, $a\notin B$ and $b\in B$. Then $(b,a)\in A\times A\subseteq A\times A\cup\mathscr{C}A\times X$, but $(b,a)\notin B\times B\cup\mathscr{C}B\times X$, a contradiction.

Corollary 2.8. *If* $\emptyset\neq B\subseteq X$ *and* $A\times A\cup\mathscr{C}A\times X\subseteq B\times B\cup\mathscr{C}B\times X$, *then* $A=B$.

Theorem 2.9. $\{O\times O\cup\mathscr{C}O\times X: O\in\mathfrak{T}\}$ *is a base for* $\mathscr{U}(\mathfrak{T})$ *iff* $\mathfrak{T}$ *consists of at most three sets.*

Proof. If $\mathfrak{T}=\{\emptyset, X\}$ or if $\mathfrak{T}=\{\emptyset, O, X\}$, then $\{X\times X\}$ or $\{O\times O\cup\mathscr{C}O\times X, X\times X\}$ is a base for $\mathscr{U}(\mathfrak{T})$.

Conversely, suppose that $\emptyset\neq O_i\neq X$ for $i=1,2$ and that $\{O\times O\cup\mathscr{C}O\times X: O\in\mathfrak{T}\}$ is a base for $\mathscr{U}(\mathfrak{T})$. Then $(O_1\times O_1\cup\mathscr{C}O_1\times X)\cap(O_2\times O_2\cup\mathscr{C}O_2\times X)\supseteq O\times O\cup\mathscr{C}O\times X$ for some $O\in\mathfrak{T}$. By Corollary 2.8, $O_1=O=O_2$ and hence $\mathfrak{T}$ consists of at most three sets.

Theorem 2.10. $(X,\mathfrak{T})$ *is disconnected iff there exists an* A *such that* $\emptyset\neq A\neq X$ *and* $A\times A\cup\mathscr{C}A\times\mathscr{C}A\in\mathscr{U}(\mathfrak{T})$.

Proof. If $(X,\mathfrak{T})$ is disconnected, let A be both open and closed and $\emptyset\neq A\neq X$. Then $A\times A\cup\mathscr{C}A\times\mathscr{C}A=(A\times A\cup\mathscr{C}A\times X)\cap(\mathscr{C}A\times\mathscr{C}A\cup A\times X)\in\mathscr{U}(\mathfrak{T})$.

Conversely, suppose that $\emptyset\neq A\neq X$ and that $A\times A\cup\mathscr{C}A\times\mathscr{C}A\in\mathscr{U}(\mathfrak{T})$. We will show that A is open (and by symmetry, $\mathscr{C}A$ is open). Let $a\in A$. Then $(A\times A\cup\mathscr{C}A\times\mathscr{C}A)[a]=A$.

Theorem 2.11. $(X,\mathscr{U}(\mathfrak{T}))$ *is totally bounded* ($U\in\mathscr{U}(\mathfrak{T})$ *implies that* $U[A]=X$ *for some finite set* A).

Proof. Let $U\in\mathscr{U}(\mathfrak{T})$. Then $U\supseteq\bigcap\{O_i\times O_i\cup\mathscr{C}O_i\times X: 1\leq i\leq n\}$. Consider the 2^n sets of the form $A_1\cap\cdots\cap A_n$ where $A_i=O_i$ or $A_i=\mathscr{C}O_i$. Pick $q\in A_1\cap\cdots\cap A_n$ whenever $A_1\cap\cdots\cap A_n\neq\emptyset$ and let A be the set of q-points thus picked. Clearly, A is finite and we show now that $U[A]$

$=X$. Let $x \in X$. Let $A_i = O_i$ if $x \in O_i$ and let $A_i = \mathscr{C} O_i$ if $x \in \mathscr{C} O_i$. Then $\bigcap A_i \neq \emptyset$. There exists a q in A such that $q \in \bigcap A_i$. Then $(q, x) \in U$ or $x \in U[A]$. If $(q, x) \notin U$, then $(q, x) \notin O_j \times O_j \cup \mathscr{C} O_j \times X$ for some j and hence $q \in O_j$ and $x \in \mathscr{C} O_j$. Then $A_j = \mathscr{C} O_j$ and $q \in A_j = \mathscr{C} O_j$, a contradiction.

Corollary 2.12. $\triangle \in \mathscr{U}(\mathfrak{T})$ *iff* (i) X *is finite and* (ii) $\mathfrak{T}$ *is discrete.*

Proof. Let $\triangle \in \mathscr{U}(\mathfrak{T})$. By Theorem 2. 11, there exists a finite set A such that $X = \triangle[A] = A$. Thus, X is finite and (i) holds. (ii) follows from the fact that $\mathscr{U}(\mathfrak{T})$ is a discrete uniform space when $\triangle \in \mathscr{U}(\mathfrak{T})$.

Conversely, suppose that (i) and (ii) hold. Then $\triangle = \bigcap\{\{x\} \times \{x\} \cup \mathscr{C}\{x\} \times X : x \in X\} \in \mathscr{U}(\mathfrak{T})$.

Theorem 2.13. *If* $\mathfrak{T}$ *is countable, then* $\mathscr{U}(\mathfrak{T})$ *has a countable base. If* $\mathscr{U}(\mathfrak{T})$ *has a countable base, then* $(X, \mathfrak{T})$ *is a second axiom space.*

Proof. If $\mathfrak{T} = \{O_i : i \in P\}$, then $\{O_i \times O_i \cup \mathscr{C} O_i \times X : i \in P\}$ is a countable subbase for $\mathscr{U}(\mathfrak{T})$ and hence $\mathscr{U}(\mathfrak{T})$ has a countable base.

Suppose $\mathscr{U}(\mathfrak{T})$ has a countable base $\{U_i : i \in P\}$. Now $U_i \supseteq \bigcap\{O_{ij} \times O_{ij} \cup \mathscr{C} O_{ij} \times X : 1 \leqq j \leqq n_i\}$ for each $i \in P$. We will show that the $\{O_{ij}\}$ forms a subbase for $\mathfrak{T}$. Let $x \in O \in \mathfrak{T}$. Then $U[x] \subseteq O$ for some $U \in \mathscr{U}(\mathfrak{T})$. But $U \supseteq U_i \supseteq \bigcap\{O_{ij} \times O_{ij} \cup \mathscr{C} O_{ij} \times X : 1 \leqq j \leqq n_i\}$ and hence $\bigcap\{O_{ij} \times O_{ij} \cup \mathscr{C} O_{ij} \times X\}[x] \subseteq O$. But $(O_{ij} \times O_{ij} \cup \mathscr{C} O_{ij} \times X)[x] = O_{ij}$ or X. Thus $x \in O^* \subseteq O$ where O^* is an intersection of sets from the collection $\{O_{ij} : 1 \leqq j \leqq n_i\}$.

Theorem 2.14. (i) *If* $(X, \mathfrak{T})$ *is regular, then* $c(\triangle) \subseteq O \times O \cup \mathscr{C} O \times X$ *for each* $O \in \mathfrak{T}$. (ii) *If* $c(\triangle) \subseteq O \times O \cup \mathscr{C} O \times X$ *for each* $O \in \mathfrak{T}$, *then* $(X, \mathfrak{T})$ *is an* R_0*-space* ($x \in O \in \mathfrak{T}$ *implies that* $c(x) \subseteq O$). (iii) *If* $(X, \mathfrak{T})$ *is* T_2 *then* $c(\triangle) \subseteq O \times O \cup \mathscr{C} O \times X$ *for each* $O \in \mathfrak{T}$.

Proof. (i) Suppose $(x, y) \notin O \times O \cup \mathscr{C} O \times X$ for some $O \in \mathfrak{T}$. Then $x \in O$ and $y \notin O$. But $x \in O^* \subseteq cO^* \subseteq O$ for some $O^* \in \mathfrak{T}$ since $(X, \mathfrak{T})$ is regular. Hence $(x, y) \in O^* \times \mathscr{C} cO^*$ and $O^* \times \mathscr{C} cO^* \cap \triangle = \emptyset$. Thus $(x, y) \notin c\triangle$.

(ii) Let $x \in O \in \mathfrak{T}$ and suppose that $c(x) \nsubseteq O$. Then take $y \in c(x) \cap \mathscr{C} O$. Thus $(x, y) \in c(x) \times c(y) \subseteq c(x) \times c(x) \subseteq c\triangle \subseteq O \times O \cup \mathscr{C} O \times X$. Hence $(x, y) \in O \times O \cup \mathscr{C} O \times X$. But $x \in O$ and $y \in \mathscr{C} O$, a contradiction.

(iii) If $(X, \mathfrak{T})$ is T_2, then $c\triangle = \triangle \subseteq O \times O \cup \mathscr{C} O \times X$ for each $O \in \mathfrak{T}$.

The converse of (i) is false; take $(X, \mathfrak{T})$ any T_2-space that is not regular. The converse of (iii) is false; take any regular space that is not

T_2. The converse of (ii) is false; take $(X, \mathfrak{T})$ an infinite space with the cofinite topology.

3. Subspaces

Theorem 3.1. *Let* $(X', \mathfrak{T}')$ *be a subspace of* $(X, \mathfrak{T})$. *Then* $\mathscr{U}(\mathfrak{T}')= X'\times X'\cap \mathscr{U}(\mathfrak{T})$.

Proof. If $O'=O\cap X'$ where $O\in\mathfrak{T}$, then $O'\times O'\cup \mathscr{C}'O'\times X'=X'\times X'\cap (O\times O\cup \mathscr{C}O\times X)$.

4. Transformations

Theorem 4.1. *Let* $(X, \mathfrak{T})$ *and* $(Y, \mathfrak{T}')$ *be topological spaces and* $f: X\to Y$ *a transformation. Then* f *is continuous relative to* $\mathfrak{T}$ *and* $\mathfrak{T}'$ *iff* f *is uniformly continuous relative to* $\mathscr{U}(\mathfrak{T})$ *and* $\mathscr{U}(\mathfrak{T}')$.

Proof. Only the necessity requires proof. Let $O'\times O'\cup \mathscr{C}O'\times Y$ be subbasic in $\mathscr{U}(\mathfrak{T}')$. Then $(f\times f)^{-1}(O'\times O'\cup \mathscr{C}O'\times Y)\supseteq(f^{-1}O'\times f^{-1}O')\cup \mathscr{C}f^{-1}O'\times X$. Since $f^{-1}O'\in\mathfrak{T}$, it follows that $(f\times f)^{-1}(O'\times O'\cup \mathscr{C}O'\times Y)\in \mathscr{U}(\mathfrak{T})$.

Theorem 4.2. *A net* $S: D\to X$ *is* $\mathscr{U}(\mathfrak{T})$*-Cauchy iff* $O\in\mathfrak{T}$ *implies that* S *is eventually in* O *or* S *is eventually in* $\mathscr{C}O$.

Proof. Let $S: D\to X$ be a $\mathscr{U}(\mathfrak{T})$-cauchy net and suppose that $O\in\mathfrak{T}$. Then there exists an N in D such that $m, n\geqq N$ implies that $(S(m), S(n))\in O\times O\cup \mathscr{C}O\times X$. Suppose S is not eventually in O nor eventually in $\mathscr{C}O$. Take $m^*\geqq N$ and $S(m^*)\notin O$. Take $n^*\geqq N$ and $S(n^*)\notin \mathscr{C}O$. Then $m^*, n^*\geqq N$, but $(S(n^*), S(m^*))\notin O\times O\cup \mathscr{C}O\times X$, a contradiction.

Conversely, suppose $S: D\to X$ is a net with the property that S is eventually in O or eventually in $\mathscr{C}O$ for each $O\in\mathfrak{T}$. We will show that S is then $\mathscr{U}(\mathfrak{T})$-cauchy. Let $O\times O\cup \mathscr{C}O\times X$ be subbasic in $\mathscr{U}(\mathfrak{T})$. If S is eventually in O, then $S\times S$ is eventually in $O\times O\subseteq O\times O\cup \mathscr{C}O\times X$. If S is eventually in $\mathscr{C}O$, then $S\times S$ is eventually in $\mathscr{C}O\times X\subseteq O\times O\cup \mathscr{C}O\times X$.

Corollary 4.3. *Let* $S: D\to X$ *be a net. Then* S *is* $\mathscr{U}(\mathfrak{T})$*-cauchy iff* S *frequently in* $O\in\mathfrak{T}$ *implies that* S *is eventually in* O.

In a space $(X, \mathfrak{T})$, a net $S: D\to X$ is called an O-net iff for $O\in\mathfrak{T}$, S frequently in O implies that S is eventually in O. In [2], the following theorem is proved.

Theorem 4.4. $(X, \mathfrak{T})$ *is compact iff every O-net in* X *converges.*

Theorem 4.5. $(X, \mathfrak{T})$ *is compact iff* $(X, \mathscr{U}(\mathfrak{T}))$ *is complete.*

Proof. $(X, \mathscr{U}(\mathfrak{T}))$ is complete iff every $\mathscr{U}(\mathfrak{T})$-cauchy net converges iff every O-net converges (Corollary 4. 3) iff $(X, \mathfrak{T})$ is compact (Theorem 4. 4).

Theorem 4.6. *Let* $f: X \to Y$ *be a transformation and* $\mathfrak{T}'$ *a topology for* Y. *Let* $\mathfrak{T}$ *be the weak topology for* X *determined by* f *and* $\mathfrak{T}'$. *Let* $\mathscr{U}$ *be the weak quasi uniformity for* X *induced by* f *and* $\mathscr{U}(\mathfrak{T}')$. *Then* $\mathscr{U} = \mathscr{U}(\mathfrak{T})$.

Proof. $f: X \to Y$ is $\mathfrak{T}$-$\mathfrak{T}'$ continuous and by Theorem 4. 1, $f: X \to Y$ is $\mathscr{U}(\mathfrak{T}) - \mathscr{U}(\mathfrak{T}')$ uniformly continuous. Thus $\mathscr{U} \subseteq \mathscr{U}(\mathfrak{T})$. We show now that $\mathscr{U}(\mathfrak{T}) \subseteq \mathscr{U}$. Let $O' \in \mathfrak{T}'$. Then $f^{-1}O' \times f^{-1}O' \cup \mathscr{C} f^{-1}O' \times X$ is subbasic in $\mathscr{U}(\mathfrak{T})$. But $f^{-1}O' \times f^{-1}O' \cup \mathscr{C} f^{-1}O' \times X \supseteq (f \times f)^{-1}(O' \times O' \cup \mathscr{C} O' \times X) \in \mathscr{U}$.

5. Products

Example 5.1. For each positive integer i, let $(X_i, \mathfrak{T}_i)$ be the two point space $\{0, 1\}$ with the discrete topology and let $(X, \mathfrak{T}) = \times\{(X_i, \mathfrak{T}_i): i \in P\}$. Then $\mathscr{U}(\mathfrak{T}) \neq \times\{\mathscr{U}(\mathfrak{T}_i): i \in P\}$. For, let $O = \cup\{P_i^{-1}[o]: i \in P\}$. Then $O \in \mathfrak{T}$ and $O \times O \cup \mathscr{C} O \times X \in \mathscr{U}(\mathfrak{T})$. But $O \times O \cup \mathscr{C} O \times X \nsupseteq (P_1 \times P_1)^{-1} \Delta \cap \cdots \cap (P_n \times P_n)^{-1} \Delta$ for every integer n and hence $O \times O \cup \mathscr{C} O \times X \notin \times\{\mathscr{U}(\mathfrak{T}_i): i \in P\}$.

REFERENCES

[1] W. J. PERVIN: Quasi Uniformization of Topological Spaces, Math. Ann. 147 (1962).
[2] L. C. ROBERTSON and S. P. FRANKLIN: O-sequences and O-nets, The American Mathematical Monthly, Vol. 72, Number 5, May 1965.

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