CONTINUITY OF ADDITIVE FUNCTIONALS

OSAMU KATSUMATA AND SHOZO KOSHI

1. Let E be a measure space with positive finite measure μ, and S be the space of all essentially finite measurble functions on E. A set X of S will be called *normal* if $x\in X$, $|x|\geqq|y|$ imply $y\in X$.

In the following, we assume X is a normal sublattice of S. A functional T defined on X is called additive if $x, y\in X$; $|x|\cap|y|=0$ imply $T(x+y)=T(x)+T(y)$.

For the integral representation of T, the continuity condition is important.

Recently, L. Drewnowski and W. Orlicz have proved that if T is (uc_∞) and (uac), then T can be represented by $f(r, t)$ such that

$$T(x)=\int_E f(x(t), t)d\mu$$

where $f(0, t)=0$ *a. e.*, $f(r, t)$ satisfies the generalized Carathéodory conditions and $f(x(t), t)$ is integrable on E for each $x\in X$. [1]

In this note, we shall prove that the continuity conditions of Drewnowski and Orlicz (cf. also Friedman and Katz [2]) is replaced by apparently weaker condition: order continuity. Here we do not assume any "uniform" property.

The proof of Theorem 7 in [4] is not sufficient, so the assumption of Theorem must be replaced by order-continuity instead of (CIII) (semicontinuity).

A sequence $x_n\in X(n=1, 2, \cdots)$ is *order-convergent* to $x\in X$, if there exists $y\in X$, $|x_n|\leqq y$ and $x_n(t)\to x(t)$ *a. e.* in E.

2. For an additive functional T on X, we shall consider the following conditions:

(uc_∞) T is *uniformly continuous* by $L_\infty(\mu)$-norm (essential supremum norm) on every set $\{y;\ |y|\leqq|x|\}$ with $x\in X$, $\|x\|_\infty<\infty$.

(uac) T is *uniformly absolutely continuous* on every set $\{y;\ |y|\leqq |x|\}$, $x\in X$; that is for each $\varepsilon>0$, there is a positive number $\delta>0$ such that $|T(y\chi_e)|\leqq\varepsilon$ for $|y|\leqq|x|$ and a measurable set e with $\mu(e)<\delta$ where χ_e is a characteristic function of e.

(o) If $x_n\in X(n=1, 2, \cdots)$ is order-convergent to $x\in X$, then $\lim_{n\to\infty} T(x_n)=T(x)$.

Theorem (o) *is equivalent to* (uc_∞) *and* (uac).

Proof. Since (uc_∞) and (uac) imply (o) obviously, we shall prove the converse.

$(o) \Rightarrow (uac)$ This can be proved by the same method used in the proof of theorem 1.3 of Drewnowski and Orlicz [1].

$(o) \Rightarrow (uc_\infty)$ It is sufficient to prove the case $X=\{x;\ |x(t)|\leqq 1$ *a. e.* $t\in E\}$. Let denote by Λ the set of all dyadic numbers in $[-1, 1]$ i. e. if $\lambda\in\Lambda$, then $\lambda=m/2^n$ for some integers $n(>0)$, m with $-2^n\leqq m\leqq 2^n$ and let $h_\lambda(t)$ $(\lambda\in\Lambda)$ be Radon-Nikodym's derivative of $T(\lambda\chi_e)$ i. e. $T(\lambda\chi_e) =\int_e h_\lambda(t)d\mu$ where χ_e is a characteristic function of a measurable set $e\subset E$. Now we take an arbitrary $\delta>0$ and fix it for a while. For integers n and $m=-2^n+1, \cdots, 0, 1, \cdots, 2^n$, we consider measurable sets:

$$e_m^n=\{t;\ |h_\lambda-h_\mu|(t)\geqq\delta \text{ for some } \lambda, \mu\in\Lambda \text{ with } (m-1)/2^n\leqq\lambda, \mu \leqq m/2^n\} \qquad (n=1, 2, \cdots;\ m=-2^n+1, \cdots, 0, 1, \cdots, 2^n)$$

and

$b_n=\bigcup_{m=-2^n+1}^{2^n} e_m^n$. Note that e_m^n is written by $e_m^n=\bigcup_{\lambda,\mu}\{t;\ h_\lambda(t)-h_\mu(t)\geqq\delta\}$ where $\lambda, \mu(\in\Lambda)$ are countable. By definition of b_n, we have $b_1\supset b_2\supset\cdots \supset b_n\supset\cdots$, and let us denote their limit $b=\bigcap_{n=1}^\infty b_n$.

(i) Let $\mu(b)=0$ for every $\delta>0$. For each $\varepsilon>0$, by (uac), there exists $\varepsilon'>0$ such that $|T(y\chi_e)|<\varepsilon$ for all $y\in X$ and for $\mu(e)<\varepsilon'$. At the same time we see that there is n with $\mu(b_n)<\varepsilon'$ such that

$$|h_\lambda(t)-h_\mu(t)|<\varepsilon \text{ for } |\lambda-\mu|\leqq 1/2^n \text{ with } \lambda, \mu\in\Lambda,\ t\in E\sim b_n.$$

This will be seen if we put $\delta=\varepsilon/2$. Let $\|x(t)-x'(t)\|_\infty<1/2^{n+1}$. By (o) we can find $\lambda_i, \mu_i\in\Lambda$ (i: finite set) and mutually disjoint measurable sets e_i such that

$$\|x-\sum_i\lambda_i\chi_{e_i}\|_\infty\leqq 1/2^{n+2}, \quad \|x'-\sum_i\mu_i\chi_{e_i}\|_\infty\leqq 1/2^{n+2}$$

and

$$|T(x)-T(\sum_i\lambda_i\chi_{e_i})|<\varepsilon, \quad |T(x')-T(\sum_i\mu_i\chi_{e_i})|<\varepsilon.$$

We have $|\lambda_i-\mu_i|<1/2^n$ for each i and

$$|T(\sum_i\lambda_i\chi_{e_i})-T(\sum_i\mu_i\chi_{e_i})|\leqq\sum_i\int_{e_i\sim b_n}|h_{\lambda_i}-h_{\mu_i}|d\mu$$

$$+|T(\sum_i \lambda_i \chi_{e_i}\chi_{b_n})|+|T(\sum_i \mu_i \chi_{e_i}\chi_{b_n})|\leqq\varepsilon\mu(E)+2\varepsilon.$$

Hence,

$$|T(x)-T(x')|\leqq|T(x)-T(\sum_i \lambda_i \chi_{e_i})|+|T(\sum_i \lambda_i \chi_{e_i})-T(\sum_i \mu_i \chi_{e_i})|$$
$$+|T(x')-T(\sum_i \mu_i \chi_{e_i})|\leqq\varepsilon\mu(E)+4\varepsilon.$$

(uc_∞) follows from this.

(ii) Let $\mu(b)>0$ for some $\delta>0$. It will be shown that this case can not occur. For every n, we shall define a partition of b by induction, that is

$$b=b^n_{-2^n+1}\cup\cdots\cup b^n_{2^n},$$

$b^n_m(m=-2^n+1,\cdots,2^n)$ are mutually disjoint measurable sets and for every $p\geqq n$, $b^n_m\subset e^{n,p}_m$ where $e^{n,p}_m=\{t\,;t\in e^p_{m'}$ for some m' with $(m-1)/2^n<m'/2^p\leqq m/2^n\}$. We take a note that the sequence $e^{n,p}_m(p=n,n+1,\cdots)$ is monotone decreasing with respect to p and that $\bigcup_{m=-2^n+1}^{2^n} e^{n,p}_m=b_p$.

Let us define $b^1_m(m=-1,0,1,2)$, at first we put

$$a^1_m=b\cap e^{1,1}_m\cap e^{1,2}_m\cap\cdots,\quad(m=-1,0,1,2).$$

Then, we have $b=\bigcup_{m=-1}^{2} a^1_m$, since $b\sim\bigcup_{m=-1}^{2} a^1_m=\bigcap_{m=-1}^{2}\bigcup_{p=1}^{\infty}(b\sim e^{1,p}_m)\subset$ $\bigcup_{p=1}^{\infty}\bigcap_{m=-1}^{2}(b\sim e^{1,p}_m)\subset\bigcup_{p=1}^{\infty}(b\sim\bigcup_{m=-1}^{2} e^{1,p}_m)=\phi$.

Hence we define, $b^1_{-1}=a^1_{-1}$, $b^1_0=a^1_0\sim a^1_{-1}$, $b^1_1=a^1_1\sim a^1_0\sim a^1_{-1}$, etc., then by definition $b^1_m\subset a^1_m\subset e^{1,p}_m$ for all $p\geqq1$. Assuming we have defined b^n_m, we shall define b^{n+1}_s; $s=-2^{n+1}+1,\cdots,0,1,\cdots,2^{n+1}$. It suffices to define b^{n+1}_{s-1}, b^{n+1}_s of the form $s=2m$ with some $m\in\{-2^n+1,\cdots,0,\cdots,2^n\}$. We set

$$b^{n+1}_s=b^n_m\cap e^{n+1,n+1}_s\cap e^{n+1,n+2}_s\cap\cdots,$$
$$b^{n+1}_{s-1}=b^n_m\sim b^{n+1}_s.$$

We must check $b^{n+1}_{s-1}\subset e^{n+1,p}_{s-1}$ for $p\geqq n+1$. This will be done, since $t\in b^{n+1}_{s-1}\Rightarrow t\notin b^{n+1}_s\Rightarrow t\notin e^{n+1,p}_s$ for some $p\geqq n+1\Rightarrow t\notin e^{n+1,k}_s$ for all $k\geqq p\Rightarrow t\in e^{n+1,k}_{s-1}$ for all $k\geqq p\Rightarrow t\in e^{n+1,p}_{s-1}$ for all $p\geqq n+1$, since $e^{n+1,k}_s\cup e^{n+1,k}_{s-1}=e^{n,k}_m$ by definition. For n, we define measurable functions $f_n, g_n\in X$ such that

$$f_n(t)=\sum_{m=-2^n+1}^{2^n}\frac{m-1}{2^n}\chi_{b^n_m}(t),$$

$$g_n(t)=\sum_{m=-2^n+1}^{2^n}\frac{m}{2^n}\chi_{b^n_m}(t).$$

We have by definition $f_n(t)\leqq f_{n+1}(t)\leqq g_{n+1}(t)\leqq g_n(t)$, and $|f_n(t)-g_n(t)|\leqq 1/2^n$. Hence, there exists $x\in X$ with $f_n\rightarrow x$, $g_n\rightarrow x$ (order convergence). For $m=-2^n+1, \cdots, 2^n$, we can decompose b^n_m into

$$b^n_m=c^n_{m,1}\cup c^n_{m,2}\cup\cdots \qquad \text{(mutually disjoint)}$$

where $c^n_{m,i}\subset\{t\,;\,h^n_{\lambda^n_{m,i}}(t)-h^n_{\mu^n_{m,i}}(t)\geqq\delta\}$ for every pair $\lambda^n_{m,i}$, $\mu^n_{m,i}\in\Lambda$ with $(m-1)/2^n\leqq\lambda^n_{m,i}$, $\mu^n_{m,i}\leqq m/2^n$, since $b^n_m\subset e^n_m=\bigcup_{\lambda,\mu}\{t\,;\,h_\lambda(t)-h_\mu(t)\geqq\delta\}$ where $(m-1)/2^n\leqq\lambda$, $\mu\leqq m/2^n$ with λ, $\mu\in\Lambda$. If we define f'_n, g'_n as follows:

$$f'_n(t)=\sum_{m=-2^n+1}^{2^n}\sum_i\lambda^n_{m,i}\chi_{c^n_{m,i}}(t)$$

$$g'_n(t)=\sum_{m=-2^n+1}^{2^n}\sum_i\mu^n_{m,i}\chi_{c^n_{m,i}}(t),$$

then $f_n(t)\leqq f'_n(t)$, $g'_n(t)\leqq g_n(t)$ and

$$T(f'_n)-T(g'_n)\geqq\delta\mu(b)$$

with $f'_n\rightarrow x$ and $g'_n\rightarrow x$ (order convergence). This contradicts to the condition (o).

Remark. In above Theorem, the condition (o) is replaced by the condition that T is continuous by measure convergence.

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DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

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