SOME GENERALIZATIONS OF DUALITY THEOREMS IN MATHEMATICAL PROGRAMMING PROBLEMS

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§ 1. Introduction and problem setting

Let X, Z and W be real linear spaces and suppose that Z and W are in duality with respect to a certain bilinear functional $((\ ,\ ))$. Let C and D be nonempty sets in X and Z respectively, and let f and g be finite-valued real functions on C and D respectively. Assume that $g(z) = -\infty$ for every $z \notin D$. Let A be a transformation from C into Z.

We shall be concerned with the following two problems:

(I) Determine $M = \inf\{f(x) - g(Ax)\,;\, x \in C\}$,

(II) Determine $M^* = \sup\{g^*(w) - f_A^*(w)\,;\, w \in W\}$,

where

$$g^*(w) = \inf\{((z, w)) - g(z)\,;\, z \in D\}$$

and

$$f_A^*(w) = \sup\{((Ax, w)) - f(x)\,;\, x \in C\}.$$

Here we define

$$r + \infty = \infty + r = \infty, \quad r - \infty = -\infty + r = -\infty$$

for all real numbers r, and set

$$\infty + \infty = \infty, \quad -\infty - \infty = -\infty, \quad -(-\infty) = \infty.$$

More precisely, we shall study the problems

(i) the existence of x or w which attains the infimum or the supremum,

(ii) relations between the values M and M^*.

An answer to problem (ii) is called a duality theorem.

R. T. Rockafellar [6] investigated these problems in the case where A is linear and continuous, C and D are convex sets and f and $-g$ are convex functions. Our problems (I) and (II) contain the problems discussed by U. Dieter [3], K. S. Kretschmer [4] and R. Van Slyke and R. Wets [7]. M. Yamasaki [8] studied the above problems in the case where C is a convex set, D is a convex cone, f is a convex function, $g=0$ and A is convex with respect to D.

In the present paper, we shall generalize duality theorems given in [3], [4], [7] and [8] by making use of a well-known separation theorem. We shall introduce in § 5 a condition which was called the normality condition in [6] and [7]. By means of this condition, duality theorems in § 3 will be generalized.

§ 2. Preliminaries

For later use, we shall recall some notions and results in [1] and [2].

Let X and Y be real linear spaces in duality with respect to a certain bilinear functional $((\ ,\ ))$. Let us denote the weak topology on X by $w(X, Y)$ and the Mackey topology by $s(X, Y)$. A locally convex Hausdorff topology $t(X, Y)$ on X compatible with this duality is stronger than $w(X, Y)$ and weaker than $s(X, Y)$. If X is assigned $t(X, Y)$, then every element of Y is identified with a $t(X, Y)$-continuous linear functional on X.

Let R be the set of real numbers and R_0 the set of non-negative real numbers.

We shall utilize the following separation theorem:

Proposition 1.[1] *Let K be a $w(X, Y)$-closed convex set in X and x_0 be an element of X such that $x_0 \notin K$. Then there exist $y_0 \in Y$ and $\alpha \in R$ such that*

$$((x_0, y_0)) > \alpha \geqq ((x, y_0))$$

for all $x \in K$.

Next we shall recall the conjugate operation of convex functions in [3], which will be used in § 4. For a finite-valued real convex function p on X with nonempty convex domain P, the conjugate function p^* and the conjugate set P^* are defined by

$$p^*(y) = \sup\{((x, y)) - p(x)\,;\, x \in P\},$$
$$P^* = \{y \in Y\,;\, p^*(y) < \infty\}.$$

Then p^* is a finite-valued real convex function with convex domain P^*.

Let us define

$$[p, P] = \{(x, r)\,;\ x \in P \text{ and } r \geqq p(x)\}.$$

For a finite-valued real concave function q on X with nonempty convex domain Q, there are similar definitions:

$$q^*(y) = \inf\{((x, y)) - q(x)\,;\, x \in Q\},$$

1) [1], p. 73, Proposition 4 and [2], p. 50, Proposition 1.

$$Q^* = \{y \in Y \,;\, q^*(y) > -\infty\},$$
$$[q, Q] = \{(x, r) \,;\, x \in Q \text{ and } r \leqq q(x)\}.$$

Then q^* is a finite-valued real concave function with convex domain Q^*.

Dieter proved

Proposition 2.[2] *Let $X \times R$ and $Y \times R$ be in duality with respect to the bilinear functional $< , >$ defined by*

$$<(x, r), (y, s)> = ((x, y)) + rs$$

for all $(x, r) \in X \times R$ and $(y, s) \in Y \times R$.

(1) *If P is $w(X, Y)$-closed and p is lower semicontinuous with respect to $w(X, Y)$, then $[p, P]$ is $w(X \times R, Y \times R)$-closed.*

(2) *If $[p, P]$ is $w(X \times R, Y \times R)$-closed, then $p^{**} = (p^*)^* = p$ and $P^{**} = (P^*)^* = P$.*

§ 3. Duality theorems

Let $Z \times R$ and $W \times R$ be in duality with respect to the bilinear functional $< , >$ defined by

$$<(z, r), (w, s)> = ((z, w)) + rs$$

for every $(z, r) \in Z \times R$ and $(w, s) \in W \times R$. Let E, E_0 and L be the sets in $Z \times R$ defined by

$$E = \{(Ax - z, r + f(x) - g(z)) \,;\, x \in C, z \in D \text{ and } r \in R_0\},$$
$$L = \{(0, r) \,;\, 0 \in Z \text{ and } r \in R\},$$
$$E_0 = E \cap L.$$

In case $C \cap A^{-1}(D)$ is not empty, we have

$$E_0 = \{(0, r + f(x) - g(Ax)) \,;\, 0 \in Z, x \in C \cap A^{-1}(D) \text{ and } r \in R_0\}.$$

First we shall study the existence of x which attains the value M of problem (I). We have

Theorem 1. *Assume that the value M is finite. Then there exists $x \in C$ such that $Ax \in D$ and $M = f(x) - g(Ax)$ if and only if the set E_0 is $w(Z \times R, W \times R)$-closed.*

Proof. Since M is finite, we have

$$\{0\} \times (M, +\infty) \subset E_0 \subset \{0\} \times [M, +\infty).$$

2) [3], p. 98, Hilfssatz 5 and p. 99, Hilfssatz 7.

Therefore the set E_0 is $w(Z\times R, W\times R)$-closed if and only if $(0, M)$ belongs to E_0. We see easily that there exists $x\in C$ such that $Ax\in D$ and $M=f(x)-g(Ax)$ if and only if $(0, M)\in E_0$.

Observe that the set E_0 is $w(Z\times R, W\times R)$-closed whenever the set E is $w(Z\times R, W\times R)$-closed, since the set L is $w(Z\times R, W\times R)$-closed. However, the $w(Z\times R, W\times R)$-closedness of the set E_0 does not necessarily imply the $w(Z\times R, W\times R)$-closedness of the set E. This is shown by Example 5. 1 in [4] or Example 3. 5 in [7].

As for the $w(Z\times R, W\times R)$-closedness of the set E_0, we have

Proposition 3. *Let X be a topological linear space and let Z be assigned $w(Z, W)$. Assume that the functions f and $-g$ are lower semicontinuous and that the transformation A is continuous. If $C\cap A^{-1}(D)$ is a nonempty and compact set, then the set E_0 is $w(Z\times R, W\times R)$-closed.*

Proof. Let $\{(0, r_t); t\in T\}$ be a net in E_0 which $w(Z\times R, W\times R)$-converges to $(z, r)\in Z\times R$. Then $z=0$ and there exists $x_t\in C\cap A^{-1}(D)$ such that $r_t\geqq f(x_t)-g(Ax_t)$. By the compactness of $C\cap A^{-1}(D)$, there exists a subnet $\{x_t; t\in T'\}$ which converges to some $x\in C\cap A^{-1}(D)$. Then by the continuity of A and the lower semicontinuity of f and $-g$, we have

$$r=\lim_{t\in T'} r_t\geqq\varliminf_{t\in T'} f(x_t)-\varlimsup_{t\in T'} g(Ax_t)\geqq f(x)-g(Ax),$$

and hence $(0, r)\in E_0$. Therefore the set E_0 is $w(Z\times R, W\times R)$-closed.

Next we shall investigate some relations between the values M and M^*. We have

Theorem 2. *It is always valid that $M^*\leqq M$.*

Proof. In case $C\cap A^{-1}(D)$ is empty, we have $M=\infty$ and our assertion is obvious. In case $C\cap A^{-1}(D)$ is not empty, let x and w be arbitrary elements of $C\cap A^{-1}(D)$ and W respectively. The inequalities

$$f(x)+f_A^*(w)\geqq((Ax, w)),$$
$$g(Ax)+g^*(w)\leqq((Ax, w))$$

follow from the definitions of f_A^* and g^* in § **1.** Thus we have

$$f(x)-g(Ax)\geqq g^*(w)-f_A^*(w).$$

This completes the proof.

Before giving the converse relation $M^*\geqq M$, we shall prepare

Lemma 1. *If* $w \in W$ *and* $\alpha \in R$ *satisfy the inequality*

$$\alpha \geqq ((u, w)) - r$$

for all $(u, r) \in E$, *then*

$$\alpha \geqq f_A^*(w) - g^*(w).$$

Proof. Since $(Ax - z,\ f(x) - g(z)) \in E$ for any $x \in C$ and $z \in D$, we have

$$\begin{aligned}\alpha &\geqq ((Ax - z, w)) - f(x) + g(z) \\ &= \{((Ax, w)) - f(x)\} - \{((z, w)) - g(z)\}.\end{aligned}$$

From the definitions of f_A^* and g^*, it follows that

$$\alpha \geqq f_A^*(w) - g^*(w).$$

Now we shall prove

Theorem 3. *If the value* M *is finite and the set* E *is convex and* $w(Z \times R,\ W \times R)$*-closed, then* $M = M^*$ *holds.*

Proof. For an arbitrarily fixed $\varepsilon > 0$, $(0, M - \varepsilon) \notin E$. Since E is a $w(Z \times R,\ W \times R)$-closed convex set, there exist $(w, s) \in W \times R$ and $\alpha \in R$ such that

$$(M - \varepsilon)s > \alpha \geqq ((u, w)) + rs$$

for all $(u, r) \in E$ by Proposition 1. From the fact that $(0, M + \varepsilon) \in E$, it follows that $(M - \varepsilon)s > (M + \varepsilon)s$ and hence $s < 0$. Writing $\alpha_0 = \alpha / s$ and $w_0 = -w/s$, we have

$$M - \varepsilon < \alpha_0 \leqq -((u, w_0)) + r$$

for all $(u, r) \in E$. By means of Lemma 1, we see that

$$\alpha_0 \leqq g^*(w_0) - f_A^*(w_0) \leqq M^*.$$

Therefore $M^* > M - \varepsilon$. By the arbitrariness of ε, we conclude that $M^* \geqq M$. The converse inequality was given in Theorem 2. This completes the proof.

Theorem 4. *If the value* M^* *is finite and the set* E *is convex and* $w(Z \times R,\ W \times R)$*-closed, then* $M = M^*$ *holds.*

Proof. Suppose $(0, M^*) \notin E$. By Proposition 1 there exist $(w, s) \in W \times R$ and $\alpha \in R$ such that

$$M^* s > \alpha \geqq ((u, w)) + rs \tag{1}$$

for all $(u, r) \in E$. For a fixed $(u_1, r_1) \in E$, we have $(u_1, r_1+t) \in E$ for all $t \in R_0$ and by (1)

$$\alpha \geqq ((u_1, w)) + r_1 s + ts.$$

Letting $t \to \infty$, we see that $s \leqq 0$. First we shall consider the case where $s < 0$. Writing $\alpha_0 = \alpha/s$ and $w_0 = -w/s$, we have

$$M^* < \alpha_0 \leqq -((u, w_0)) + r \tag{2}$$

for all $(u, r) \in E$. It follows from Lemma 1 that

$$\alpha_0 \leqq g^*(w_0) - f_A^*(w_0) \leqq M^*.$$

This is a contradiction. Next we shall consider the case where $s = 0$. Then we have

$$0 > \alpha \geqq ((u, w)) \tag{3}$$

for all $(u, r) \in E$. On the other hand, there exist $v \in W$ and $\beta \in R$ such that

$$\beta \geqq ((u, v)) - r \tag{4}$$

for all $(u, r) \in E$. In fact, by our assumption that M^* is finite, we can find $v \in W$ such that both $f_A^*(v)$ and $g^*(v)$ are finite. By the definitions of f_A^* and g^*, we have

$$\beta = f_A^*(v) - g^*(v) \geqq ((Ax, v)) - f(x) - ((z, v)) + g(z)$$
$$\geqq ((Ax - z, v)) - \{r + f(x) - g(z)\}$$

for all $x \in C$, $z \in D$ and $r \in R_0$, which implies (4). On account of (3) and (4), we have

$$\alpha t + \beta \geqq ((u, tw + v)) - r$$

for all $(u, r) \in E$ and $t \in R_0$. We see by Lemma 1 that

$$\alpha t + \beta \geqq f_A^*(tw + v) - g^*(tw + v) \geqq -M^*.$$

Letting $t \to \infty$, we have $M^* = \infty$, since $\alpha < 0$. This is a contradiction. Thus $(0, M^*) \in E$. It follows that $M^* \geqq M$. On account of Theorem 2, we have $M = M^*$.

With regard to the convexity of the set E, we have

Theorem 5. *Assume that* C *and* D *are convex sets and* f *and* $-g$ *are convex functions. If any one of the following conditions* (M. 1) *and* (M. 2) *is fulfilled, then the set* E *is convex*:

(M. 1) *A is linear,*

(M. 2) *D is a cone, A is convex with respect to D*[3], *i. e.,*

$$A(tx_1+(1-t)x_2)-tAx_1-(1-t)Ax_2\in D$$

for any $x_1, x_2\in C$ and $t\in R_0$ with $0<t<1$, and g is increasing with respect to D, i. e., $g(z_1)\geqq g(z_2)$ whenever $z_1-z_2\in D$.

Proof. Assume condition (M. 2). Let $(u_i, r_i)\in E(i=1, 2)$ and $t\in R_0$, $0<t<1$. Then there exist $x_i\in C, z_i\in D$ and $s_i\in R_0$ such that $u_i=Ax_i-z_i$ and $r_i=s_i+f(x_i)-g(z_i)$. Let us denote $u_t=tu_1+(1-t)u_2$, $r_t=tr_1+(1-t)r_2$, $x_t=tx_1+(1-t)x_2$, $z_t=tz_1+(1-t)z_2$ and $s_t=ts_1+(1-t)s_2$. Then $x_t\in C$, $z_t\in D$ and $s_t\in R_0$. Since A is convex with respect to D, we have $tAx_1+(1-t)Ax_2=Ax_t-v$ for some $v\in D$. Thus $u_t=Ax_t-(v+z_t)\in A(C)-D$. On the other hand, by the convexity of f and $-g$ and by the assumption that g is increasing with respect to D, we have

$$\begin{aligned} r_t&=s_t+tf(x_1)+(1-t)f(x_2)-tg(z_1)-(1-t)g(z_2)\\ &\geqq s_t+f(x_t)-g(z_t)\geqq s_t+f(x_t)-g(v+z_t), \end{aligned}$$

and hence $r_t=s+f(x_t)-g(v+z_t)$ for some $s\in R_0$. Therefore $(u_t, r_t)\in E$ and the set E is convex. Similarly we can prove that condition (M. 1) implies the convexity of the set E.

By means of Theorem 5, we see that Theorems 3 and 4 are some generalizations of duality theorems in [3], [4], [7] and [8].

We shall study the $w(Z\times R, W\times R)$-closedness of the set E. In the rest of this section, we always assume that X is a topological linear space, that Z is assigned $w(Z, W)$, that the sets C and D are closed, that the functions f and $-g$ are lower semicontinuous and that the transformation A is continuous. Then we have

Theorem 6. *Assume that, for any $w(Z\times R, W\times R)$-convergent net $\{(u_t, r_t); t\in T\}$ in E, there exist $\{x_t; t\in T\}\subset C$ and $\{z_t; t\in T\}\subset D$ such that*

$$u_t=Ax_t-z_t,\ r_t\geqq f(x_t)-g(z_t)$$

and $\{x_t; t\in T\}$ contains a convergent subnet. Then the set E is $w(Z\times R, W\times R)$-closed.

Proof. Let $\{(u_t, r_t); t\in T\}$ be a net in E which $w(Z\times R, W\times R)$-converges to $(u, r)\in Z\times R$. By our assumption, there exist $\{x_t; t\in T\}\subset C$ and $\{z_t; t\in T\}\subset D$ such that

$$u_t=Ax_t-z_t,\ r_t\geqq f(x_t)-g(z_t)$$

3) We correct the definition of this notion in [8], p. 332 in the present form.

and $\{x_t; t\in T\}$ contains a subnet $\{x_t; t\in T'\}$ which converges to some x. Then $\{z_t; t\in T'\}$ converges to $Ax-u=z$, since A is continuous. Since C and D are closed, we have $x\in C$ and $z\in D$. By the lower semicontinuity of f and $-g$, we have

$$r=\lim_{t\in T'} r_t \geqq \underline{\lim}_{t\in T'} f(x_t)-\overline{\lim}_{t\in T'} g(z_t)\geqq f(x)-g(z).$$

Therefore $(u, r)\in E$ and the set E is $w(Z\times R, W\times R)$-closed.

Corollary. *If the set C is compact, then the set E is $w(Z\times R, W\times R)$-closed.*

Similarly we can prove

Proposition 4. *Assume that A is homeomorphic and that the set D is compact. Then the set E is $w(Z\times R, W\times R)$-closed.*

§ 4. The case where A is linear and continuous

We shall recall the convex programming problems studied by Rockafellar [6].

Let X and Y be real linear spaces which are in duality with respect to the bilinear functional $((\ ,\ ))_1$ and let Z and W be real linear spaces which are in duality with respect to the bilinear functional $((\ ,\ ))_2$. Let C and D be nonempty convex sets in X and Z respectively, and let f and $-g$ be finite-valued real convex functions on C and D respectively. Let A be a linear transformation from X into Z which is $w(X, Y)-w(Z, W)$ continuous and let A^* be its adjoint. Thus A^* is a linear transformation from W into Y which is $w(W, Z)-w(Y, X)$ continuous and satisfies $((Ax, w))_2=((x, A^*w))_1$ for all $x\in X$ and $w\in W$.

By virtue of the conjugate operations for convex sets and convex functions defined in § 2, we see that the function g^* defined in § 1 is the conjugate function of the concave function g and that $f_A^*(w)=f^*(A^*w)$ holds, where f^* is the conjugate function of the convex function f. Let us denote by C^* and D^* the conjugate sets of convex sets C and D respectively. The convex programming problems discussed in [6] are as follows:

(III) Determine $N=\inf\{f(x)-g(Ax); x\in C \text{ and } Ax\in D\}$,

(IV) Determine $N^*=\sup\{g^*(w)-f^*(A^*w); w\in D^* \text{ and } A^*w\in C^*\}$.

Here we use the convention that the infimum and the supremum on the empty set are equal to $+\infty$ and $-\infty$ respectively.

These problems contain the problems investigated by Dieter [3], Kretschmer [4]. Dieter discussed the case where $X=Z$ and A is the identity transformation. Kretschmer discussed the case where

$$f(x)=((x, y_0))_1, \quad C=P,$$
$$g(z)=0, \qquad D=Q+z_0,$$

where P and Q are convex cones which are $w(Z, W)$-closed and $w(Z, W)$-closed respectively, and $y_0 \in Y$ and $z_0 \in Z$ are fixed elements. In this case, problems (III) and (IV) are called linear programming problems. Van Slyke and Wets [7] investigated problem (III) in the case where $g=0$ and $D=\{b\}$, $(b \in Z)$.

Now we shall apply our results in § 3 to problems (III) and (IV). On account of Theorems 3, 4 and 5, we have

Proposition 5. *Let $Z \times R$ and $W \times R$ be in duality as in* § 3 *and let E be the set in $Z \times R$ defined by*

$$E=\{(Ax-z, r+f(x)-g(z))\,;\, x \in C, z \in D \text{ and } r \in R_0\}.$$

If the set E is $w(Z \times R, W \times R)$-closed and either N or N^* is finite, then $N=N^*$ holds.

Since problems (III) and (IV) have symmetry, we can derive a dual statement to the above result. Observing that

$$-N^*=\inf\{(-g^*(w))-(-f^*(A^*w))\,;\, w \in D^* \text{ and } A^*w \in C^*\},$$

we shall consider the following problem:

(V) Determine $-N^{**}=\sup\{-f^{**}(x)+g^{**}(A^{**}x)\,;\, x \in C^{**}$ and $A^{**}x \in D^{**}\}$.

It is always valid that $N^{**} \leqq N$. If the sets $[f, C]$ and $[g, D]$ defined in § 2 are $w(X \times R, Y \times R)$-closed and $w(Z \times R, W \times R)$-closed respectively, then $f^{**}=f$, $g^{**}=g$, $C^{**}=C$ and $D^{**}=D$ by Proposition 2. In this case, the set F in $Y \times R$ defined by

$$F=\{(A^*w-y, r-g^*(w)+f^*(y))\,;\, w \in D^*, y \in C^* \text{ and } r \in R_0\}$$

plays the role of the set E in § 3. Noting $A^{**}=A$ and applying Theorems 3, 4 and 5, we have

Proposition 6. *Assume that the sets $[f, C]$ and $[g, D]$ are $w(X \times R, Y \times R)$-closed and $w(Z \times R, W \times R)$-closed respectively. If the set F is $w(Y \times R, X \times R)$-closed and either N or N^* is finite, then $N=N^*$ holds.*

We shall give an application of Theorem 6.

Proposition 7. *Let C and D be $w(X, Y)$-closed and $w(Z, W)$-closed respectively and let f and $-g$ be lower semicontinuous with respect to $w(X, Y)$ and $w(Z, W)$ respectively. Assume that any $w(X, Y)$-bounded set in X is relatively $w(X, Y)$-compact. If we further assume that $A^*(D^*)\cap(C^*)^\circ$ is not empty, then the set E is $w(Z\times R, W\times R)$-closed, where $(C^*)^\circ$ denotes the $s(Y, X)$-interior of C^*.*

Proof. Let $\{(u_t, r_t); t\in T\}$ be a net in E which $w(Z\times R, W\times R)$-converges to $(u, r)\in Z\times R$. Then there exist $x_t\in C$ and $z_t\in D$ such that $u_t=Ax_t-z_t$ and $r_t\geqq f(x_t)-g(z_t)$. By the definitions of f^* and g^*, we have

$$r_t\geqq((x_t, y))_1-((z_t, w))_2-f^*(y)+g^*(w)$$

for all $y\in C^*$ and $w\in D^*$. By our assumption, there are y_0 and w_0 such that $w_0\in D^*$ and $y_0=A^*w_0\in(C^*)^\circ$. For any $y\in Y$, there exists $\varepsilon>0$ such that $y_0\pm\varepsilon y\in(C^*)^\circ$. Consequently

$$\begin{aligned} r_t&\geqq((x_t, y_0\pm\varepsilon y))_1-((z_t, w_0))_2-f^*(y_0\pm\varepsilon y)+g^*(w_0)\\ &=((Ax_t-z_t, w_0))_2\pm\varepsilon((x_t, y))_1-f^*(y_0\pm\varepsilon y)+g^*(w_0)\\ &=((u_t, w_0))_2\pm\varepsilon((x_t, y))_1-f^*(y_0\pm\varepsilon y)+g^*(w_0). \end{aligned}$$

Since $\{r_t-((u_t, w_0))_2; t\in T\}$ converges to $r-((u, w_0))_2$, there is $t_0\in T$ such that $\{r_t-((u_t, w_0))_2; t\in T, t>t_0\}$ is bounded. Consequently $\{((x_t, y))_1; t\in T, t>t_0\}$ is bounded for every $y\in Y$, and hence $\{x_t; t\in T, t>t_0\}$ is relatively $w(X, Y)$-compact by our assumption. Thus $\{x_t; t\in T\}$ contains a $w(X, Y)$-convergent subnet. Therefore the set E is $w(Z\times R, W\times R)$-closed by Theorem 6.

Note that any $w(X, Y)$-bounded set in X is relatively $w(X, Y)$-compact provided that Y is a disk space (= espace tonnelé) and X is the topological dual space of Y ([2], p. 65, Théorème 1).

§ 5. Normality condition

We return to the general problem (I). Let E be as defined in § 3 and denote by $\bar{E}$ the $w(Z\times R, W\times R)$-closure of E. We shall introduce another quantity m defined by

$$m=\inf\{r; r\in R \text{ and } (0, r)\in\bar{E}\},$$

where we set $m=\infty$ in the case where $(0, r)\notin\bar{E}$ for any $r\in R$. This quantity was called the subvalue in the case of linear programming problems (cf. [4]).

We have

Theorem 7. *It is always valid that* $M^* \leqq m \leqq M$.

Proof. The inequality $m \leqq M$ follows immediately from the definitions of m and M. To prove $M^* \leqq m$, we may suppose that $m < \infty$. Let $(0, r) \in \bar{E}$. Then there exists a net $\{(u_t, r_t); t \in T\}$ in E which $w(Z \times R, W \times R)$-converges to $(0, r)$. For every $t \in T$, there exist $x_t \in C$, $z_t \in D$ and $s_t \in R_0$ such that $u_t = Ax_t - z_t$ and $r_t = s_t + f(x_t) - g(z_t)$. By the definitions of f_A^* and g^*, we have

$$r_t \geqq f(x_t) - g(z_t) \geqq ((Ax_t, w)) - f_A^*(w) - ((z_t, w)) + g^*(w)$$
$$= ((u_t, w)) + g^*(w) - f_A^*(w)$$

for any $w \in W$ and hence $r \geqq g^*(w) - f_A^*(w)$. Thus we have $M^* \leqq m$.

Theorem 8. *If the set* $\bar{E}$ *is convex and* $M^* > -\infty$, *then* $M^* = m$ *holds.*

Proof. On account of Theorem 7, it suffices to show the inequality $M^* \geqq m$ in the case where M^* is finite. Suppose $(0, M^*) \notin \bar{E}$. Applying Proposition 1 to $(0, M^*)$ and the $w(Z \times R, W \times R)$-closed convex set $\bar{E}$, we can arrive at a contradiction by the same argument as in the proof of Theorem 4. Therefore $(0, M^*) \in \bar{E}$. Thus we have $M^* \geqq m$.

Theorem 9. *If the set* $\bar{E}$ *is convex and* $m < \infty$, *then* $M^* = m$ *holds.*

Proof. By Theorem 7, it is enough to show the inequality $M^* \geqq m$ in the case where m is finite. For an arbitrarily fixed $\varepsilon > 0$, we have $(0, m - \varepsilon) \notin \bar{E}$ by the definition of m. Applying Proposition 1 to $(0, m - \varepsilon)$ and the $w(Z \times R, W \times R)$-closed convex set $\bar{E}$, we can prove the inequality $m - \varepsilon < M^*$ by the same argument as in the proof of Theorem 3. By the arbitrariness of ε, we have $m \leqq M^*$.

Note that the set $\bar{E}$ is convex whenever the set E is convex ([1], p. 50, Proposition 14).

By means of Theorem 5, we see that Theorems 8 and 9 are some generalizations of Theorem 2 in [4].

Now we introduce

Definition. Problem (I) is said to be normal if $\bar{E} \cap L = \bar{E}_0$, where L and E_0 are the sets defined in § 3 and $\bar{E}_0$ is the $w(Z \times R, W \times R)$-closure of E_0.

The normality condition was first introduced in [6], cf. [7].

We shall prove

Theorem 10. *Problem* (I) *is normal if and only if* $M=m$.

Proof. Observe that $\bar{E}_0=\{0\}\times[M,+\infty)$ in case M is finite, $\bar{E}_0=L$ in case $M=-\infty$ and $\bar{E}_0$ is empty in case $M=\infty$. Similarly, $\bar{E}\cap L=\{0\}\times[m,+\infty)$ in case m is finite, $\bar{E}\cap L=L$ in case $m=-\infty$ and $\bar{E}\cap L$ is empty in case $m=\infty$. Our theorem follows from these obsevations.

From Theorems 8, 9 and 10, we obtain

Corollary 1. *Assume that problem* (I) *is normal and* $\bar{E}$ *is convex. If* $M<\infty$ *or* $-\infty<M^*$, *then* $M=M^*$.

From Theorems 7 and 10, we have

Corollary 2. *If* $M=M^*$, *then problem* (I) *is normal.*

These corollaries are a generalization of Theorem 7 of [6].

We easily have

Proposition 8. *If the set* E *is* $w(Z\times R, W\times R)$*-closed, then problem* (I) *is normal.*

By this proposition, we see that Corollary 1 is a generalization of Theorems 3 and 4. However it seems difficult to verify the normality in the case where the set E is not $w(Z\times R, W\times R)$-closed.

We have

Proposition 9. *Assume that the set* E *is convex. If* L *intersects the* $s(Z\times R, W\times R)$*-interior* E° *of* E, *then problem* (I) *is normal.*

Proof. Suppose $\bar{E}_0\neq\bar{E}\cap L$. Then there exists $(0,r_1)\in\bar{E}\cap L$ such that $(0,r_1)\notin\bar{E}_0$. By our assumption, there is $(0,r_0)\in L$ such that $(0,r_0)\in E^\circ$. Let U be a convex $s(Z\times R, W\times R)$-neighborhood of $(0,r_0)$ satisfying $U\subset E^\circ$. Since $(0,r_1)\in\bar{E}$ and E is convex, we see that the set

$$V=\{(z,s)\,;\,z=tu,\ s=(1-t)r_1+tr \text{ for all } (u,r)\in U \text{ and } t\in R_0 \text{ with } 0<t\leqq 1\}$$

is contained in E ([1], p. 51, Proposition 15). It is clear that $L\cap V\subset L\cap E=E_0$. Since $(0,(1-t)r_1+tr_0)\in L\cap V$ for all t, $0<t\leqq 1$, we see that $(0,r_1)\in\overline{L\cap V}\subset\bar{E}_0$. This is a contradiction. Therefore $\bar{E}_0=\bar{E}\cap L$.

This is a straightforward extension of Proposition 5.2 in [7].

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