A RADICAL AND A SUBCATEGORY IN AN EXACT CATEGORY

Dedicated to Professor Keizo Asano on his 60th birthday

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0. Introduction

In [1], Amitur presented an axiomatic treatment of radicals of rings. Let π be a ring property. Then N is called a *π-radical* of a ring R if N is a unique maximal π-ideal of R and R/N is π-semisimple. The main properties which a radical is generally required to satisfy are: the existence in every ring; the radical of an ideal I should be the intersection of I and the radical of the whole ring. Up to now various radicals of rings has been treated. In the category theory, Maranda [4] introduced the notion of radical as a subfunctor T of the identity such that $T(A/T(A))=0$ for every object A in a category $\mathscr{A}$. Occasionally, a radical (functor) possesses some properties, for example; T is an *idempotent* radical (if T is a radical and $T^2=T$), T is a *torsion* radical or *hereditary* radical (if T is a radical and left exact).

In **1**, we construct a radical functor from a π-property and give the condition that this radical functor is an idempotent, hereditary radical. In 2, according to Miyashita [6], we give an analogue to the usual theorem on the Jacobson radical of a ring in an exact category and in **3**, we consider the special subcategory with respect to the radical functor given in 2. For notations and terminologies which are not introduced here we refer the readers to [5].

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1. Radical property

Let $\mathscr{A}$ be a category with kernels and cokernels. $\mathscr{A}$ is called an *exact* category provided that

(i) every monomorphism is a kernel of some morphism in $\mathscr{A}$,
(i*) every epimorphism is a cokernel of some morphism in $\mathscr{A}$,
(ii) every morphism in $\mathscr{A}$ is a composite of a monomorphism and an

epimorphism. (Note that the decomposition is unique up to equivalence.)

Let $\mathscr{A}$ be an arbitrary category. If $f: A \longrightarrow B$ is a morphism and $A' \longrightarrow B$ is a monomorphism, we shall denote the image of the composition $A' \longrightarrow A \xrightarrow{f} B$ by $f(A')$ without confusion. A is called the subobject of B if there is a monomorphism from A to B, where A, $B \in \mathrm{Obj}\, \mathscr{A}$, and we shall refer to $A \longrightarrow B$ as the inclusion. We define the suboject $A_1 \longrightarrow B$ to be contained in $A_2 \longrightarrow B$ if there is a morphism $A_1 \longrightarrow A_2$ such that $A_1 \longrightarrow B = A_1 \longrightarrow A_2 \longrightarrow B$. (Note that $A_1 \longrightarrow A_2$ must be a monomorphism and unique.) In this case we write $A_1 \subseteq A_2 (\subseteq B)$. If A_1 and A_2 are subobjects of A such that $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$, then A_1 and A_2 are isomorphic subojects of A. Therefore we may regard A_1 and A_2 as the same subobject.

An exact category $\mathscr{A}$ has images and inverse images and finite intersections and finite unions (see [5; Prop. 1. 14. 1, p. 18 and Cor. 1. 15. 3]).

Now let π be a subobject property. For any A in a category with unions, we denote by $\pi(A)$ the class of all π-subobjects of A. Then we set $\bigcup_i A_i = R(A)$, where A_i runs through $\pi(A)$. We define the following:

Definition 1. 1. Let $\mathscr{A}$ be an exact category with unions. We will call the property π a *radical* property if the following conditions are satisfied:

(1) Zero object 0 is a π-subobject of every $A \in \mathrm{Obj}\, \mathscr{A}$.

(2) Every image of a π-subobject is a π-subobject.

(3) $A/R(A)$ contains no non-zero π-subobject.

Throughout the rest of this section, $\mathscr{A}$ is a locally small exact category with unions unless otherwise specified.

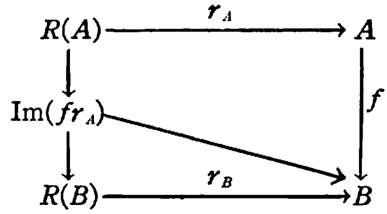
Now let $\mathscr{R}$ be the full subcategory of $\mathscr{A}$ with the object class $\{R(A)\}_{A \in \mathrm{Obj}\mathscr{A}}$, and $r_A: R(A) \longrightarrow A$ be the inclusion morphism in $\mathscr{A}$. For any $f: A \longrightarrow B$ in $\mathscr{A}$, we have

$$\mathrm{Im}(A_i \longrightarrow R(A) \xrightarrow{r_A} A \xrightarrow{f} B) \subseteq R(B)$$

(where $A_i \in \pi(A)$) by the condition (2). Since $\mathscr{A}$ is an exact category with unions, we have

$$(*) \qquad f(R(A)) = f(\textstyle\bigcup_i A_i) = \bigcup_i f(A_i) \subseteq R(B).$$

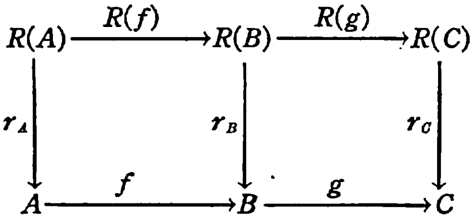
by [5; Prop. 1. 11. 2]. Hence we have the following commutative diagram:

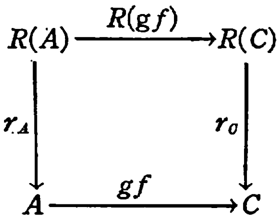


where $R(A) \longrightarrow \mathrm{Im}(fr_A) \longrightarrow B$ is the decomposition of $R(A) \xrightarrow{r_A} A \xrightarrow{f} B$ and $\mathrm{Im}(fr_A) \longrightarrow R(B)$ is given by (*). In the above diagram, the morphism $R(A) \longrightarrow R(B)$ is uniquely induced by f, which will be denoted by $R(f)$.

Theorem 1. 1. *$R: \mathscr{A} \longrightarrow \mathscr{R}$ is a subfunctor of the identity.*

Proof. Consider the following two commutative diagrams in $\mathscr{A}$:





Then we have

$$r_C R(g) R(f) = g f r_A = r_C R(gf).$$

Since r_C is a monomorphism, $R(g)R(f) = R(gf)$. On the other hand, Im $(1_A r_A) = \mathrm{Im}(r_A) = R(A)$, and so we have $R(1_A) = 1_{R(A)}$.

Corollary 1. 2. *R is a radical functor in the sense of Dickson* [2] *and Maranda* [4].

Proof. By the radical property (3), $A/R(A)$ does not have non-zero π-subobject. Therefore $R(A/R(A)) = 0$.

Consider an arbitrary category $\mathscr{A}$, a subcategory $\mathscr{A}'$, and $A \in \mathrm{Obj}\mathscr{A}$. A *reflection* for $A \in \mathrm{Obj}\mathscr{A}$ is an $\mathrm{Ref}(A) \in \mathrm{Obj}\mathscr{A}'$ together with a morphism

$$p_A : \mathrm{Ref}(A) \longrightarrow A$$

such that for every $A' \in \mathrm{Obj}\mathscr{A}'$ and every morphism $A' \longrightarrow A$ in $\mathscr{A}$ there exists a unique morphism $A' \longrightarrow \mathrm{Ref}(A)$ in $\mathscr{A}'$ making the diagram

$$\begin{array}{ccc} A' & & \\ \downarrow & \searrow & \\ \mathrm{Ref}(A) & \xrightarrow{p_A} & A \end{array}$$

commutative. If every object in $\mathscr{A}$ has a reflection in $\mathscr{A}'$, then $\mathscr{A}'$ is called a *reflective* subcategory of $\mathscr{A}$, and Ref becomes a functor from $\mathscr{A}$ to $\mathscr{A}'$, called the *reflector* of $\mathscr{A}$ in $\mathscr{A}'$. Then we have:

Theorem 1. 3. *The full subcategory $\mathscr{R}$ is a reflective subcategory of $\mathscr{A}$ with reflector R if and if R is an idempotent radical.*

Proof. Let $\mathscr{R}$ is a reflective subcategory with reflector R. Then there is a unique morphism $R(A) \longrightarrow R(R(A))$ such that the diagram

$$\begin{array}{ccc} R(A) & & \\ \downarrow & \searrow^{1_{R(A)}} & \\ R(R(A)) & \xrightarrow{r_{R(A)}} & R(A) \end{array}$$

is commutative. Clearly $R(A) = R(R(A))$, that is, R is idempotent.

Conversely, if R is idempotent, that is, $R(B) = R(R(B))$ for all $B \in \mathrm{Obj}\mathscr{A}$, then $R(B)$ is the union of all π-subobjects of $R(B)$. By the condition (2) and [5; Prop. 1. 11. 2], we have the commutative diagram

$$\begin{array}{ccc} R(B) & & \\ \downarrow & \searrow & \\ R(A) & \xrightarrow{r_A} & A \end{array}$$

for any morphism $R(B) \longrightarrow A$ in $\mathscr{A}$, where $R(B) \longrightarrow R(A)$ is the inclusion through $\mathrm{Im}(R(B) \longrightarrow A)$. Hence $\mathscr{R}$ is a reflective subcategory of $\mathscr{A}$ with reflector R.

Theorem 1. 4. *R is hereditary if and only if for any monomorphism $f: A \longrightarrow B$, $R(A) = A \cap R(B)$.*

Proof. Consider the commutative diagram in $\mathscr{A}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(A) & \xrightarrow{R(f)} & R(B) & \xrightarrow{R(g)} & R(C) & & \\ & & \downarrow{\scriptstyle r_A} & & \downarrow{\scriptstyle r_B} & & \downarrow{\scriptstyle r_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array} \tag{1. 5}$$

where $g: B \longrightarrow C$ is the cokernel of f. Then we have a monomorphism $fr_A = r_B R(f)$ and therefore $R(f)$ is a monomorphism. On the other hand $R(g)R(f) = R(gf) = 0$, and $\mathrm{Ker}(R(g))$ contains $\mathrm{Im}(R(f))$.

Now let $X \xrightarrow{x} R(B) \xrightarrow{R(g)} R(C) = 0$. Since $0 = r_C R(g)x = gr_B x$, there is a morphism $a: X \longrightarrow A$ such that $r_B x = fa$. By [5; Prop. 1. 8. 1], $R(A) = R(B) \cap A$ if and only if the left square in (1. 5) is a pull back diagram. Therefore, if $R(A) = R(B) \cap A$, there is a morphism $X \longrightarrow R(A)$ such that

$$X \longrightarrow R(A) \longrightarrow R(B) = X \longrightarrow R(B).$$

Hence we have $\mathrm{Ker}(R(g)) = R(A)$.

Conversely, if R is a left exact functor, we have a row exact commutative diagram (1. 5). Then for any morphisms x and y where

$$X \xrightarrow{x} R(B) \xrightarrow{r_B} B = X \xrightarrow{y} A \xrightarrow{f} B,$$

we obtain $0 = gfy = gr_B x = r_C R(g)x$. r_C is a monomorphism, so $R(g)x = 0$. Then there is a morphism $X \longrightarrow R(A)$ such that $X \longrightarrow R(A) \longrightarrow R(B) = X \longrightarrow R(B)$. Therefore we get

$$X \longrightarrow R(A) \longrightarrow A \longrightarrow B = X \longrightarrow A \longrightarrow B.$$

But $A \longrightarrow B$ is a monomorphism, so we have

$$X \longrightarrow R(A) \longrightarrow A = X \longrightarrow A.$$

Hence the left square of the diagram (1. 5) is a pull back diagram, that is,

$$R(A) = R(B) \cap A.$$

2. Special redical

In this section, we consider a special π-property.

Definition 2. 1. Let M be an object in a category $\mathscr{A}$. A subobject K of M in $\mathscr{A}$ is said to be *superfluous* in M if for any subobject A of M with the inclusion $v: A \longrightarrow M$, $K \cup A = M$ implies that v is an isomorphism. ($A = M$ in the sense of 1.) This is equivalent to $K \cup A \neq M$ for any proper subobject A. (Proper implies that v is not an isomorphism.) Evidently zero object 0 is superfluous in M.

The following follows [3; p. 362].

Lemma 2. 1. *Let M be an object in an exact category $\mathscr{A}$ and let $S(M)$ be the class of all subobject of M. If A, B, C are in $S(M)$ such that $B \subseteq A$, then*

$$A \cap (B \cup C) = B \cup (A \cap C).$$

Proposition 2. 2. *Let $\mathscr{A}$ be an exact category and M, $M' \in \mathrm{Obj}\mathscr{A}$. If K is a superfluous subobject of M then so is $f(K)$ of M' for all $f \in [M, M']_{\mathscr{A}}$, where $[M, M']_{\mathscr{A}}$ is the set of all morphisms from M to M' in $\mathscr{A}$.*

Now let $\mathscr{A}$ be a locally small exact category with unions and $R^*(M)$ the union of all superfluous subobjects in M. Then by the Proposition 2. 2, we have:

Corollary 2. 3. *In a locally small exact category $\mathscr{A}$ with unions, the following follows:*

(1) *$f(R^*(M)) \subseteq R^*(M)$ for any $f \in [M, M]_{\mathscr{A}}$.*

(2) *$R^*(A) \subseteq R^*(M)$ for any subobject A of M.*

If $\{A_i\}$ is an arbitrary ascending chain of subobjects of A and if B is any subobject of A such that B is contained in $\cup_i A_i$, then B is contained in A_i for some i. Under this condition, we have:

Theorem 2. 4. *Let $\mathscr{A}$ be a locally small exact category with unions and intersections satisfying the condition as above. Then the following follows:*

(i) *If $R^*(M) \neq M$, then M has a maximal subobject. In this case, $R^*(M)$ is the intersection of all maximal subobjects of M.*

(ii) *$R^*(M/R^*(M)) = 0$.*

Proof. (i): Assume $R^*(M) \neq M$. Let K be any subobject of M with $K \not\subseteq R^*(M)$. Clearly K is not a superfluous subobject of M. Then there is a proper subobject X of M such that $K\cup X=M$. Now we set $S=\{X_i : X_i$ is a subobject of M containing X and $X_i\not\supset K\}$. Then using Zorn's lemma, X is contained in some maximal subobject X_0 of M with $K\not\subset X_0$. This shows that $R^*(M)$ contains the intersection of all maximal subobjects of M. Evidently every superfluous subobject of M is contained in all maximal subobjects of M. Hence $R^*(M)$ coincides the intersection of all maximal subobjects of M.

(ii): If $M=R^*(M)$, then $R^*(M/R^*(M))=0$ is clear. If $R^*(M)\neq M$, then M has a maximal subobject A. Since A contains $R^*(M)$, there is a one to one correspondence between maximal subobjects of M and maximal subobjects of $M/R^*(M)$. Therefore the intersection of all maximal subobjects of $M/R^*(M)$ is equal to $R^*(M)/R^*(M)=0$ and this intersection contains $R^*(M/R^*(M))$. Hence $R^*(M/R^*(M)=0$.

Now let π be, "*being superfluous*", then under the assumption of Theorem 2. 4, π is a radical property by Proposition 2. 2 and Theorem 2. 4. Hence R^* is a radical functor. $R^*(M)$ exactly corresponds to the Jacobson radical of a ring. But it is not known whether in Theorem 2. 4, if M has a maximal subobject then $R^*(M)$ is the intersection of all maximal subobjects of M. In general, this radical functor R^* is not left exact. Throughout the rest of this section, $\mathscr{A}$ is a locally small exact category.

Let Perf $(\mathscr{A})$ be the full subcategory of $\mathscr{A}$ with all $M\in\mathrm{Obj}\mathscr{A}$ which satisfies the following condition: For any subobject A of M, there is a subobject B such that $A\cup B=M$ and $A\cap B$ is superfluous in B. We call B a *s-complement* of A (in M), and Perf $(\mathscr{A})$ the *perfect full subcategory* of $\mathscr{A}$. Often, a s-complement of A will be denote by A_s though it is not unique. Then the following is essentially given in [6; Prop. 1. 3], or will be shown by direct computation:

Proposition 2. 5. *Let* $M=A\cup B\in\mathrm{Obj}\mathscr{A}$, *where* A *and* B *are subobjects of* M. *Then the following holds*:

(i) B *is a s-complement of* A *if and only if* B *is minimal subobject of* M *with respect to the property* $A\cup B=M$.

(ii) *If* B *is a s-complement of* A, *then* $K\cap B$ *is superfluous in* B *for every superfluous subobject* K *of* M.

The next will be easily seen in [6; Lemma 1. 5]:

Lemma 2. 6. *Let* A, B, C *be subobjects of* $M\in\mathrm{Obj}\mathscr{A}$ *such that* $A\cup C=B\cup C$ *and* $A\cap C=B\cap C$. *If* $A\supseteq B$, *then* $A=B$.

Theorem 2. 7. *Let M be an object in $\mathscr{A}$ with unions. If a subobject A of M satisfies the descending chain condition for subobjects and $A\cap R^*(M)=0$, then A is the union $\bigcup_{i=1}^n A_i$ of a finite number of minimal subobjects A_i of A such that $A_i\cap A_j=0$ for $i\neq j$.*

Proof. By the Proposition 2. 5 (i) and Lemma 2. 6, A has a s-complement subobject B in M. Since $A\cap B\subseteq R^*(B)\subseteq R^*(M)$ by Corollary 2. 3, we have $A\cap B\subseteq A\cap R^*(M)=0$. Hence $M=A\cup B$. Since A satisfies the descending chain condition for subobjects and $R^*(A)=R^*(M)\cap A=0$, the above argument shows that for any subobject A' of A, we have $A'\cup A'_s$ $=A$ and $A'\cap A'_s=0$. Repeating the above argument, by the descending chain condition, we shall obtain our conclusion.

Corollary 2. 8. *If M satisfies the descending chain condition for subobject and $R^*(M)=0$, then M is the union $\bigcup_{i=1}^n M_i$ of a finite number of minimal subobjects M_i of M such that $M_i\cap M_j=0$ for $i\neq j$.*

3. Subcategory Perf ($\mathscr{A}$)

In this section, $\mathscr{A}$ be a locally small exact category with unions and we consider the full subcategory Perf ($\mathscr{A}$) given in 2. The following Propositions are essentially given in [6 ; p. 89]:

Propsition 3. 1. *If M is an object in Perf ($\mathscr{A}$), then M/K is an object in Perf ($\mathscr{A}$) for any superfluous subobject K in M.*

Proposition 3. 2. *Let $0\longrightarrow B\longrightarrow M$ be an exact sequence in $\mathscr{A}$. If M is an object in Perf ($\mathscr{A}$), then so is B.*

Theorem 3. 3. Perf ($\mathscr{A}$) *is an exact full subcategory of $\mathscr{A}$.*

Proof. Let $A\longrightarrow M$ is a monomorphism in Perf ($\mathscr{A}$). If B is a s-complement of A in M, then B is in Perf ($\mathscr{A}$) by Proposition 3. 2 and $A\cap B$ is superfluous by the definition. Therefore $B/(A\cap B)=M/A$ is an Perf ($\mathscr{A}$) by [5 ; Cor. 1. 16. 7] and Proposition 3. 1. Since Perf ($\mathscr{A}$) is full subcategory, the exact sequence

$$0\longrightarrow A\longrightarrow M\longrightarrow M/A\longrightarrow 0$$

(in $\mathscr{A}$) is in Perf ($\mathscr{A}$). Hence Perf ($\mathscr{A}$) is a normal subcategory with cokernels. By Proposition 3. 2, Perf ($\mathscr{A}$) is a conormal subcategory with kernels. Since Perf ($\mathscr{A}$) is full subcategory of the exact category $\mathscr{A}$, the exactness is clear by Proposition 3. 2.

In general, the familiar distributivity relation for sets

$$(**)\qquad (\cup_i A_i)\cap B = \cup_i (A_i \cap B)$$

does not hold in a category. However, if we assume that $\{A_i\}$ is a direct family of subgroups of an abelian group A and B is another subgroup, then it can be seen that (**) holds. Finally we shall prove the following:

Theorem 3. 4. *Let $\mathscr{A}$ be a locally small exact category with unions and intersections. Assume for any direct family of subobjects $\{A_i\}$ and any subobject B of A,* (**) *holds. If R^*, introduced in* 2, *is a radical functor, then R^* is a left exact functor from* Perf ($\mathscr{A}$) *to $\mathscr{P}$.*

Proof. Let $0 \longrightarrow A \xrightarrow{f} M \xrightarrow{g} B \longrightarrow 0$ be an arbitrary exact sequence in Perf ($\mathscr{A}$). By Theorem 1. 3, R^* is the left exact functor if and only if $R^*(A) = R^*(M)\cap A$. Therefore it is sufficient to show that $R^*(A)$ contains $R^*(M)\cap A$. Since $R^*(M)$ is the union of all superfluous subobject K_i of M, then we have,

$$R^*(M)\cap A = (\cup_i K_i)\cap A = \cup_i (K_i \cap A)$$

by the condition (**). Since M is in Perf ($\mathscr{A}$), A is a s-complement of some subobject A' of M. Then $K_i \cap A$ is superfluous in A by Proposition 2. 5 (ii). Hence $K_i \cap A$ is contained in $R^*(A)$ and so $R^*(M)\cap A$ is contained in $R^*(A)$.

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