SOME REMARKS ON ASYMPTOTES IN A METRIC SPACE

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In the paper we prove three theorems on a G-space defined by H. Busemann [1]*). The definition of a G-space and the notations in the paper are the same as in [1]. The definition of an asymptote is a little modified by following Y. Nasu [2]. We first prove a theorem with respect to the divergence property in [1] (p. 230), i. e., we give a condition under which this property holds in a straight G-space of 2-dimensions, where the word "*dimension*" is given in Menger Uryson's sense. Next we deal with the set of the asymptotic conjugate points to a ray, i. e., we prove two theorems. One is concerned with the metric of a 2-dimensional G-space such that the set $K(\mathfrak{r})$ of the asymptotic conjugate points to a ray $\mathfrak{r}$ and the other is concerned with the connected components of the set $K(\mathfrak{r})$. The results are shown in the theorems 1, 2 and 3.

§ **1.** We prove the following

Theorem 1. *Let $\mathfrak{R}$ be a straight G-space of 2-dimensions and p an arbitrary point of $\mathfrak{R}$. Further let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be two rays issuing from the point p and $x_1(t)$, $0\leqq t<+\infty$, and $x_2(t)$, $0\leqq t<+\infty$, be the parametric representations by their arc-lengths of the rays $\mathfrak{r}_1$ and $\mathfrak{r}_2$ respectively. If for a point p, there exists a positive number ε such that for any two rays $\mathfrak{r}_1$ and $\mathfrak{r}_2$ issuing from the point p a positive number t_0 exists such that $x_1(t)\mathfrak{r}_2>\varepsilon$ for $t\geqq t_0$ (or $x_2(t)\mathfrak{r}_1>\varepsilon$ for $t\geqq t_0$), then the divergence property holds in the space.*

Proof. Let q be a point on the ray $\mathfrak{r}_1$ such that $x_1(t)=q$ and r a foot of the point q on the ray $\mathfrak{r}_2$ such that $x_2(t')=r$. If t' is always bounded as t tends to infinity, then the condition of the theorem holds and we see

$$\lim_{t\to\infty} x_1(t)\mathfrak{r}_2=\infty \qquad (1)$$

If t tends to infinity and then t' also tends to infinity but the segment

*) Numbers in brackets refer to the references at the end of the paper.

$T(q, r)$ always contains a bounded point, the property (1) holds again. Hence it is necessary to prove that the property (1) (or $\lim_{t\to\infty} x_2(t)\mathfrak{r}_1=\infty$) holds in the case where $T(q, r)$ is always unbounded as t tends to infinity.

The point p has a sphere neighborhood $S(p, \gamma)$ $(\gamma>0)$ such that $S(p, \gamma)$ is divided into two domains D_1 and D_2, since the space is a simply connected 2-dimensional manifold [1]. Let r_1 and r_2 be points on $\mathfrak{r}_1$ and $\mathfrak{r}_2$ respectively such that r_1, $r_1 \varepsilon S(p, \gamma)$, and further let C_1 and C_2 be simple curves which connect r_1 and r_2 such that the domains D_1 and D_2 contain C_1 and C_2 except the end points r_1 and r_2. We take $n+1$ points $q_1^0(=r_1)$, q_1^1, q_1^2, $\cdots$, $q_1^n(=r_2)$, and $q_2^0(=r_1)$, q_2^1, $\cdots$, $q_2^n(=r_2)$ on the curves C_1 and C_2 respectively. Let $\mathfrak{r}_1^0(=\mathfrak{r}_1)$, $\mathfrak{r}_1^1$, $\cdots$, $\mathfrak{r}_1^{n-1}$, $\mathfrak{r}_1^n(=\mathfrak{r}_2)$ and $\mathfrak{r}_2^0(=\mathfrak{r}_1)$, $\mathfrak{r}_2^1$, $\cdots$, $\mathfrak{r}_2^{n-1}$, $\mathfrak{r}_2^n(=\mathfrak{r}_2)$ be the rays issuing from the point p through the points q_1^0, q_1^1, $\cdots$, q_1^n and q_2^0, q_2^1, $\cdots$, q_2^n respectively. Then for two consecutive rays $\mathfrak{r}_i^j$ and $\mathfrak{r}_i^{j+1}$ where $i=1, 2$, $j=0, 1, \cdots, n-1$, a positive number t_i^j exists such that

$$x_1^j(t)\mathfrak{r}_1^{j+1}>\varepsilon \text{ for } t\geqq t_1^j \tag{2}$$

$$(\text{or } x_2^j(t)\mathfrak{r}_2^{j+1}>\varepsilon \text{ for } t\geqq t_2^j),$$

where $x_i^j(t)$, $0\leqq t<+\infty$, are the parametric representations by arc-lengths $(i=1, 2,\ j=0, 1, \cdots, n-1)$. This easily follows from the assumption of the theorem.

We put

$$t_0=max(t_i^j,\ i=1, 2,\ j=0, 1, \cdots, n-1).$$

We then have $2n$ inequalities: $x_i^j(t)\mathfrak{r}_i^{j+1}>\varepsilon$ for $t\geqq t_0$ $i=1, 2$, $j=0, 1, \cdots, n-1$. Further we have from the above assumption that $T(q, r)$ cuts the rays $\mathfrak{r}_1^0$, $\mathfrak{r}_1^1$, $\cdots$, $\mathfrak{r}_1^n$ (or $\mathfrak{r}_2^0$, $\mathfrak{r}_2^1$, $\cdots$, $\mathfrak{r}_2^n$) at points s_1^0, s_1^1, $\cdots$, s_1^n (or s_2^0, s_2^1, $\cdots$, s_2^n) respectively such that

$$s_i^j=x_i^j(u_i^j),\quad i=1, 2,\quad j=0, 1, \cdots, n,$$

$$s_1^0=s_2^0=q,\quad s_1^n=s_2^n=r,\ \text{and}\ u_i^j>t_0.$$

On the other hand, it is easy to see $s_i^j s_i^{j+1}\geqq\varepsilon$ for any $i=1, 2$, $j=0, 1, \cdots, n-1$. We thus have

$$T(q, r)>n\varepsilon.$$

Since n is an arbitrary positive integer, it is easy to see that the property (1) holds. Thus the theorem is proved.

§ 2. In this paragraph we prove two theorems. Before we do it, let

$\mathfrak{r}$ be a ray and $\mathfrak{x}$ a coray from a point p to the ray $\mathfrak{r}$. The carrier of all corays to the ray $\mathfrak{r}$ which contain $\mathfrak{x}$ as a subray is called the asymptote through the point p to the ray $\mathfrak{r}$. If the asymptote has a point a as its initial point, a is said an asymptotic conjugate point to the ray $\mathfrak{r}$. There does not necessarily exist only one asymptote from the point a to the ray $\mathfrak{r}$. Now we prove the following

Theorem 2. *Let $\mathfrak{R}$ be a G-space and $\mathfrak{r}$ a ray on $\mathfrak{R}$. If the set of asymptotic conjugate points to the ray $\mathfrak{r}$ has an isolated point p, then the corays issuing from the point p simply cover the whole space except the point p. If the space is of 2-dimensions in Menger Uryson's sense and a positive number τ_0 exists such that $\mathfrak{R}-\overline{S(p, \tau_0)}$ is of non-positive curvature, $\mathfrak{R}-\overline{S(p, \tau_0)}$ is a non-expanding tube.*

Remarks. The definition of non-positive curvature is given in [1]. Under the condition of the theorem the whole space is not of non-positive curvature. If the space is of non-positive curvature the set of asymptotic conjugate points to a ray is vacous.

Proof. Let p be an isolated point of the set $K(\mathfrak{r})$. Then p has a neighborhood $S(p, \tau)(\tau>0)$ disjoint from the set $K(\mathfrak{r})$ except the point p. If a point x of $S(p, \tau)$ does not coincide with the point p, there exists a unique coray $\mathfrak{x}_x$ from x to the ray $\mathfrak{r}$. The asymptote $\mathfrak{A}_x$ to the ray $\mathfrak{r}$, which contains $\mathfrak{x}_x$ as a subray, has p as its initial point. We show this.

If this is not so, the initial point of $\mathfrak{A}_x$ is not csntained in $S(p, \tau)$. Let $\{p_n\}$ be a sequence of points in $S(p, \tau)$ which converges to the point p and $\mathfrak{A}_{p_n}$ the asymptote to the ray $\mathfrak{r}$ through p_n. Suppose each $\mathfrak{A}_{p_n}$ has not its initial point in $S(p, \tau)$. Then a suitable subsequence of $\{\mathfrak{A}_{p_n}\}$ converges to a coray to the ray $\mathfrak{r}$ through the point p, which contradicts that p is an asymptotic conjugate point. If τ is sufficiently small, the initial point of an asymptote through a point of $S(p, \tau)$ coincides with the point p. It follows from this that all rays issuing from the point p simply cover the whole space and the set $K(\mathfrak{r})$ coincides with the point p. Thus the first part of the theorem is proved. Next we prove the last part.

It is easy to see that the limit circle at the point p to the ray $\mathfrak{r}$ is the point p itself. It follows from this that the limit circles to the ray $\mathfrak{r}$ coincide with the circles whose centers are the point p. The relation between ray and coray with respect to the rays issuing from the point p is symmetric and transitive in $\mathfrak{R}-\overline{S(p, \tau)}$ and so is on the space $\mathfrak{R}$. It is also clear that the asymptote containing $\mathfrak{r}$ as a subray has p as its initial point. The point p is also the set of the asymptotic conjugate points to any ray issuing from p.

Since $\mathfrak{R}-\overline{S(p, \tau_0)}$ is a half open tube, the universal covering surface $\widetilde{\mathfrak{R}}'$ is homeomorphic to a half open plane whose boundary lies over the circle $C(p, \tau_0)$ $(=x \mid px=\tau_0)$. Let $\tilde{\mathfrak{r}}$ and $\tilde{\mathfrak{r}}'$ be consecutive rays on $\widetilde{\mathfrak{R}}'$ whose initial points are $\tilde{q}$ and $\tilde{q}'$ respectively. Then $\tilde{q}$ and $\tilde{q}'$ lie over the initial point q of the ray $\mathfrak{r}$ and the boundary curve from $\tilde{q}$ to $\tilde{q}'$ lie over the circle $C(p, \tau_0)$. We denote this curve by $\widetilde{C}$. Let r and s be points on $C(p, \tau_0)$ sufficiently near q such that the point q lies between r and s, and suppose $\mathfrak{x}_r$ and $\mathfrak{x}_s$ are corays to $\mathfrak{r}$ from the points r and s respectively. Then we can see that there exist the rays $\tilde{\mathfrak{x}}_r$ and $\tilde{\mathfrak{x}}_s$ which lie over the rays $\mathfrak{x}_r$ and $\mathfrak{x}_s$ respectively and issue from the points $\tilde{r}$ and $\tilde{s}$ on $\widetilde{C}$ which lie over the points r and s respectively. The rays $\tilde{\mathfrak{x}}_r$ and $\tilde{\mathfrak{x}}_s$ are corays to one of the rays $\tilde{\mathfrak{r}}$ and $\tilde{\mathfrak{r}}'$. To prove the last part of the theorem, we assume that the half open tube $\mathfrak{R}-\overline{S(p, \tau_0)}$ is an expanding one. Then the rays $\tilde{\mathfrak{r}}$ and $\tilde{\mathfrak{r}}'$ are not corays on $\widetilde{\mathfrak{R}}'$ each other. Now we can further assume that $\tilde{\mathfrak{x}}_r$ is a coray to the ray $\tilde{\mathfrak{r}}$ and $\tilde{\mathfrak{x}}_s$ a coray to the ray $\tilde{\mathfrak{r}}'$. Then by the assumption the points $\tilde{r}$ and $\tilde{s}$ are near the end points $\tilde{q}$ and $\tilde{q}'$ of $\widetilde{C}$ respectively. It is easy to see that $\widetilde{C}$ contains a point which lies over an asymptotic conjugate point lying on the circle $C(p, \tau_0)$. This contradicts that the circle $C(p, \tau_0)$ is disjoint from the set of the asymptotic conjugate points. We see from this that the half open tube $\mathfrak{R}-\overline{S(p, \tau_0)}$ is non-expanding one. The theorem is thus proved.

Next we prove the following

Theorem 3. *Let $\mathfrak{R}$ be a G-space, $\mathfrak{r}$ a ray on $\mathfrak{R}$ and the set $K(\mathfrak{r})$ of the asymptotic conjugate points to $\mathfrak{r}$. If the set $K(\mathfrak{r})$ is closed and compact, the set $K(\mathfrak{r})$ is connected.*

Proof. To prove the theorem, suppose the set $K(\mathfrak{r})$ consists of two components K_1 and K_2. Then the sets K_1 and K_2 are closed and disjoint each other. We see from this that there exist two open sets O_1 and O_2 such that $O_1 \supset K_1$, $O_2 \supset K_2$ and $O_1 \cap O_2 = \phi$ and further such that the asymptote through any point of O_i $(i=1, 2)$ has a point of K_i $(i=1, 2)$ as its initial point. Let C be a simple curve from a point p_1 of K_1 to a point p_2 of K_2. When a point p moves from p_1 to p_2 on C, if p is near p_1, the initial point of the asymptote through p is a point of K_1 and, if p is near p_2, the initial point of the asymptote through p is a point of

K_2. It follows from this that a point p exists on C such that the asymptote through p is straight. If this is not so, this contradicts that the sets K_1 and K_2 are disjoint each other. On the other hand, if there exists the asymptote through p which is a straight line, we again come to a contradiction, since the sets K_1 and K_2 are compact. It follows from this that there can not exist on $\mathfrak{R}$ two compact components of the set $K(\mathfrak{r})$. The theorem is thus proved.

As can easily be seen from the theorem, if the set $K(\mathfrak{r})$ is closed and not connected, any component of the set $K(\mathfrak{r})$ is not compact. Similarly the set $K(\mathfrak{r})$ does not contain a compact component and an unbounded component.

Example. In a 3-dimensional Euclidean space referred to the rectangular coordinates (x, y, z), consider an ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ $(a>c>b>0)$. We then have a surface S by joining the half upper part of the ellipsoid: $z=\sqrt{1-x^2/a^2-y^2/b^2}$ and the half cylinder: $x^2/a^2+y^2/b^2=1$, $z\leqq 0$. The section of the ellipsoid by xz-plane is a geodesic curve. We denote this section by C. The point $A(a, 0, 0)$ has a neighborhood V such that $\bar{V}\cap C$ is the shortest connection between the end points. Thus we see that there exist on C the points A' and A'' such that the subarc of C from A' and A'' is the shortest connection with A as its center and the largest among such subarcs. In this case, the points A' and A'' are symmetric with respect to xy-plane. Similarly there exist on the ellipsoid the points B' and B'' which have the above property and have $B(-a, 0, 0)$ as its center. Then the subarc of C from A' to B' is supposed to be on the half ellipsoid of the surface S. We denote this arc by K. The arc K is compact and the set of asymptotic conjugate points to the ray: $x=a$, $y=0$, $-\infty<z\leqq 0$.

REFERENCES

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