ON UNIFORMITIES GENERATED BY FILTERS

NORMAN LEVINE

1. A filter $\mathfrak{F}$ on a set X generates a uniformity $\mathscr{U}(\mathfrak{F})$ by taking as base sets of the form $\Delta\cup F\times F$ where Δ is the diagonal in $X\times X$ and $F \in\mathfrak{F}$.

In § **4**, we obtain characterizations for principal ultrafilters, principal filters, filters $\mathfrak{F}$ with the property that $\cap\mathfrak{F}$ has at most one point.

In § **5**, we characterize the topologies which arise from uniformities generated by filters.

Completeness, total boundedness and compactness of $\mathscr{U}(\mathfrak{F})$ are treated in § **6**, **7**, and **8**.

We shall call a uniformity $\mathscr{U}$ for a set X a filter generated uniformity if there exists a filter $\mathfrak{F}$ on X such that $\mathscr{U}=\mathscr{U}(\mathfrak{F})$. Such uniformities will be termed fg-uniformities. In this case, $(X, \mathscr{U})$ will be called an fg-uniform space, or simply an fg-space.

In § **9**, we show that subspaces and quotient spaces of fg-spaces are fg-spaces. Furthermore, the supremum of a family of fg-uniformities is an fg-uniformity.

2. Theorem 2. 1 *Let $\mathfrak{F}$ be a filter on X and let $\mathscr{U}(\mathfrak{F})$ be the set of relations U such that $X\times X\supseteq U\supseteq\Delta\cup F\times F$ for some F in $\mathfrak{F}$. Then $\mathscr{U}(\mathfrak{F})$ is a uniformity for X.*

Proof. (i) $\Delta\subseteq\Delta\cup F\times F$ (ii) $(\Delta\cup F\times F)^{-1}=\Delta\cup F\times F$ (iii) $(\Delta\cup F\times F) \cap(\Delta\cup F'\times F')=\Delta\cup(F\cap F')\times(F\cap F')$ and (iv) $(\Delta\cup F\times F)\circ(\Delta\cup F\times F)=\Delta \cup F\times F$.

3. Theorem 3. 1 *Let $\mathfrak{F}_1\subseteq\mathfrak{F}_2$ be two filters on X. Then $\mathscr{U}(\mathfrak{F}_1)\subseteq \mathscr{U}(\mathfrak{F}_2)$.*

The proof is trivial.

Frequent use will be made of the following

Lemma 3. 2 *Let F and F^* be two subsets of X and suppose that F^* has at least two elements. If $\Delta\cup F\times F\supseteq\Delta\cup F^*\times F^*$, then $F\supseteq F^*$.*

Proof. Let $x\in F^*$. Take $y\neq x$ and $y\in F^*$. Then $(x, y)\in F^*\times F^* \subseteq\Delta\cup F\times F$. Thus $(x, y)\in F\times F$ and $x\in F$.

Theorem 3. 3 *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two filters on X and suppose that $\mathfrak{F}_2$ is not a principal ultrafilter. If $\mathscr{U}(\mathfrak{F}_1)\subseteq\mathscr{U}(\mathfrak{F}_2)$, then $\mathfrak{F}_1\subseteq\mathfrak{F}_2$.*

Proof. Let $F_1\in\mathfrak{F}_1$. Then $\Delta\cup F_1\times F_1\in\mathscr{U}(\mathfrak{F}_1)$ and hence $\Delta\cup F_1\times F_1\supseteq\Delta\cup F_2\times F_2$ for some $F_2\in\mathfrak{F}_2$. Since $\mathfrak{F}_2$ is not a principal ultrafilter, F_2 has at least two elements and hence by lemma 3.2 $F_1\supseteq F_2$. Thus $F_1\in\mathfrak{F}_2$.

4. In this paragraph, we obtain characterizations for various kinds of filters $\mathfrak{F}$ in terms of the associated uniformity $\mathscr{U}(\mathfrak{F})$.

Theorem 4. 1 *A filter $\mathfrak{F}$ on X is a principal ultrafilter iff $\mathscr{U}(\mathfrak{F})$ is discrete.*

Proof. Suppose $\mathscr{U}(\mathfrak{F})$ is discrete. Then $\Delta\supseteq\Delta\cup F\times F$ for some $F\in\mathfrak{F}$. Clearly then, F is a singleton set and $\mathfrak{F}$ is a principal ultrafilter. Conversely, suppose that $\mathfrak{F}$ is a principal ultrafilter. Then there exists a point $x\in X$ such that $\{x\}\in\mathfrak{F}$. Then $\Delta=\Delta\cup\{x\}\times\{x\}$ and hence $\Delta\in\mathscr{U}(\mathfrak{F})$. Thus $\mathscr{U}(\mathfrak{F})$ is discrete.

Corollary 4. 2 *A filter $\mathfrak{F}$ on X is a principal filter iff $\mathscr{U}(\mathfrak{F})$ is a principat filter.*

Proof. Case 1. Suppose $\mathfrak{F}$ is a principal ultrafilter. Then $\mathscr{U}(\mathfrak{F})$ is discrete by theorem 4.1 and hence is a principal filter. Case 2. $\mathfrak{F}$ is not a principal ultrafilter. Suppose that $\mathscr{U}(\mathfrak{F})$ is a principal filter. Then there exists an $F^*\in\mathfrak{F}$ such that $U\supseteq\Delta\cup F^*\times F^*$ for all $U\in\mathscr{U}(\mathfrak{F})$. Then $\Delta\cup F\times F\supseteq\Delta\cup F^*\times F^*$ for all $F\in\mathfrak{F}$ and by lemma 3.2, $F\supseteq F^*$ since F^* has at least two points. Thus $\mathfrak{F}$ is a principal filter. Conversely, suppose $\mathfrak{F}$ is a principal filter. Then there exists an $F^*\in\mathfrak{F}$ such that $F\supseteq F^*$ for all $F\in\mathfrak{F}$. Then $\Delta\cup F\times F\supseteq\Delta\cup F^*\times F^*$ for all $F\in\mathfrak{F}$. It follows then that $\mathscr{U}(\mathfrak{F})$ is a principal filter.

Theorem 4. 3 *Let $\mathfrak{F}$ be a filter on the set X. Then $\cap\mathfrak{F}$ has at most one point iff $\mathscr{U}(\mathfrak{F})$ is separated.*

Proof. Suppose that $x\neq y$ and $\{x, y\}\subseteq\cap\mathfrak{F}$. Then $(x, y)\in\Delta\cup F\times F$ for all $F\in\mathfrak{F}$. Then $(x, y)\in U$ for all $U\in\mathscr{U}(\mathfrak{F})$ and thus $\Delta\neq\cap\mathscr{U}$. Thus $\mathscr{U}$ is not separated. Conversely, in $\mathscr{U}$ is not separated, then $\Delta\neq\cap\mathscr{U}$. Then there exist points $x\neq y$ such that $(x, y)\in\cap\mathscr{U}$. Thus, $(x, y)\in\Delta\cup F\times F$ for all $F\in\mathfrak{F}$ and hence $\{x, y\}\subseteq F$ for all $F\in\mathfrak{F}$. It follows then that $\cap\mathfrak{F}$ contains more than one point.

Corollary 4. 4 *Let $\mathfrak{F}$ be a filter on X. Then $\mathfrak{F}$ is a principal ultrafilter or $\bigcap\mathfrak{F}=\Phi$ iff $\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$ is discrete.*

Proof. If $\mathfrak{F}$ is a principal ultrafilter, then $\mathscr{U}(\mathfrak{F})$ is discrete by theorem 4.1. Thus $\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$ is discrete. If $\bigcap\mathfrak{F}=\Phi$, then $\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$ is discrete. For let $x\in X$. Then $x\notin F$ for some F in $\mathfrak{F}$. Hence $(\varDelta\cup F\times F)[x]=\{x\}$ and thus $\{x\}$ is open. Conversely, suppose that $\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$ is discrete and suppose that $\mathfrak{F}$ is not a principal ultrafilter. We will show that $\bigcap\mathfrak{F}=\Phi$. Let $x\in X$. There exists an $F^*\in\mathfrak{F}$ such that $(\varDelta\cup F^*\times F^*)[x]=\{x\}$. Since $\mathfrak{F}$ is not a principal ultrafilter, F^* has at least two points. If $x\in F^*$, then $(\varDelta\cup F^*\times F^*)[x]=F^*\neq\{x\}$, a contraciction. It follows then that $\bigcap\mathfrak{F}=\Phi$.

5. **Theorem 5. 1** *Let $(X,\mathfrak{J})$ be a topological space. Then there exists a filter $\mathfrak{F}$ on X for which $\mathfrak{J}=\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$ iff there exist sets A and B in X such that* (1) $X=A\cup B$, (2) *$a\in A$ implies that $\{a\}$ is both open and closed,* (3) *$\mathscr{N}(b_1)=\mathscr{N}(b_2)$ for all b_1 and b_2 in B, $\mathscr{N}$ denoting neighborhood system and* (4) $A\cap B=\Phi$.

Proof. Suppose that there exists a filter $\mathfrak{F}$ on X such $\mathfrak{J}=\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$. Case 1. $\mathfrak{F}$ is a principal ultrafilter. Then by theorem 4.1, $\mathscr{U}(\mathfrak{F})$ is discrete and thus $\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$ is discrete. Let $A=X$ and $A=\Phi$. Clearly, (1)—(4) hold. Case 2. $\mathfrak{F}$ is not a principal ultrafilter. In this case, let $B=\bigcap\mathfrak{F}$ and $A=\mathscr{C}B$, $\mathscr{C}$ denoting the complement operator. Clearly, (1) and (4) hold. We show now that (2) holds. If a is in A, then $a\notin F^*$ for some F^* in $\mathfrak{F}$. Then $(\varDelta\cup F^*\times F^*)[a]=\{a\}$ and thus $\{a\}$ is both open and closed. To show (3), let $b\in B$. We will show that $\mathscr{N}(b)=\mathfrak{F}$. If $N\in\mathscr{N}(b)$, there exists an $F\in\mathfrak{F}$ such that $(\varDelta\cup F\times F)[b]\subseteq N$. Since $b\in\bigcap\mathfrak{F}$, it follows that $F\subseteq N$ and hence $N\in\mathfrak{F}$. Conversely, let $F\in\mathfrak{F}$. Then $(\varDelta\cup F\times F)[b]=F$ and hence $F\in\mathscr{N}(b)$.

Conversely, suppose that there exist sets A and B in X for which (1)—(4) hold. Case 1. $B=\Phi$. Then $\mathfrak{J}$ is discrete and $\mathfrak{J}=\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$ where $\mathfrak{F}$ is any principal ultrafilter. Case 2. $B\neq\Phi$. Let $\mathfrak{F}=\mathscr{N}(b)$ where b is arbitrary in B. We assert first that $\mathfrak{J}\subseteq\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$. To this end, let $x\in O\in\mathfrak{J}$. If $x\in A$, then $x\neq b$ and $b\in\mathscr{C}\{x\}\in\mathscr{N}(b)$. Thus $\mathscr{C}\{x\}=F\in\mathfrak{F}$ for some F. It follows then that $\{x\}=(\varDelta\cup F\times F)[x]\subseteq O$. If $x\in B$, then $\mathscr{N}(x)=\mathscr{N}(b)=\mathfrak{F}$ and hence $O\in\mathfrak{F}$. Thus $(\varDelta\cup O\times O)[x]=O$. We show next that $\mathfrak{J}(\mathscr{U}(\mathfrak{F}))\subseteq\mathfrak{J}$. For let $x\in O\in\mathfrak{J}(\mathscr{U}(\mathfrak{F}))$. If $x\in A$, then $x\in\{x\}\subseteq O$ and $\{x\}\in\mathfrak{J}$. If $x\in B$, then $\mathscr{N}(x)=\mathscr{N}(b)=\mathfrak{F}$. But there exists an $F\in\mathfrak{F}$ such that $(\varDelta\cup F\times F)[x]\subseteq O$ and hence $x\in F\subseteq O$. Since

$F \in \mathscr{N}(x)$, it follows that $O \in \mathfrak{F}$.

6. Lemma 6. 1 *$\mathfrak{F}$ is a $\mathscr{U}(\mathfrak{F})$-cauchy filter in X.*

Proof. If $U \in \mathscr{U}(\mathfrak{F})$, then $U \supseteq \Delta \cup F \times F \supseteq F \times F$ for some $F \in \mathfrak{F}$.

Lemma 6. 2 *If $\mathfrak{F}^*$ is a $\mathscr{U}(\mathfrak{F})$ cauchy filter in X and if $\mathfrak{F}^*$ is not a principal ultrafilter, then $\mathfrak{F}^* \supseteq \mathfrak{F}$.*

Proof. Let $F \in \mathfrak{F}$. Then $\Delta \cup F \times F \in \mathscr{U}(\mathfrak{F})$ and hence $F^* \times F^* \subseteq \Delta \cup F \times F$ for some F^* in $\mathfrak{F}^*$. But F^* has at least two points since $\mathfrak{F}^*$ is not a principal ultrafilter. It follows from lemma 3.2 that $F^* \subseteq F$ and $F \in \mathfrak{F}^*$.

Theorem 6. 3 *Suppose $\mathfrak{F}$ is a filter on X with the propery that $\cap \mathfrak{F} = \emptyset$. Then $\mathfrak{F}$ is an ultrafilter iff $\mathfrak{F}$ is the only $\mathscr{U}(\mathfrak{F})$-cauchy filter which is not principal.*

Proof. Suppose that $\mathfrak{F}$ is an ultrafilter. Since $\cap \mathfrak{F} = \emptyset$, $\mathfrak{F}$ is not principal and by lemma 6.1, $\mathfrak{F}$ is $\mathscr{U}(\mathfrak{F})$-cauchy. Suppose now that $\mathfrak{F}^*$ is any $\mathscr{U}(\mathfrak{F})$-cauchy, non principal filter. By lemma 6.2, $\mathfrak{F}^* \supseteq \mathfrak{F}$ and since $\mathfrak{F}$ is an ultrafilter, it follows that $\mathfrak{F}^* = \mathfrak{F}$. Thus $\mathfrak{F}$ is the only $\mathscr{U}(\mathfrak{F})$-cauchy non principal filter on X.

Conversely, suppose that $\mathfrak{F}$ is the only $\mathscr{U}(\mathfrak{F})$-cauchy, non principal filter on X. To show that $\mathfrak{F}$ is an ultrafilter, let $\mathfrak{F}' \supseteq \mathfrak{F}$. Then $\cap \mathfrak{F}' \subseteq \cap \mathfrak{F} = \emptyset$ and thus $\mathfrak{F}'$ is not principal. $\mathfrak{F}'$ is clearly $\mathscr{U}(\mathfrak{F})$-cauchy since $\mathfrak{F}$ is. Thus $\mathfrak{F}' = \mathfrak{F}$.

7. Completeness of $\mathscr{U}(\mathfrak{F})$ is investigated in this paragraph.

Theorem 7. 1 *$(X, \mathscr{U}(\mathfrak{F}))$ is complete iff $\mathfrak{F}$ is a convergent filter.*

Proof. If $(X, \mathscr{U}(\mathfrak{F}))$ is complete, then $\mathfrak{F}$ is convergent since by lemma 6.1, $\mathfrak{F}$ is $\mathscr{U}(\mathfrak{F})$-cauchy. Conversely, suppose that $\mathfrak{F}$ is convergent and that $\mathfrak{F}^*$ is a $\mathscr{U}(\mathfrak{F})$-cauchy filter. Case 1. $\mathfrak{F}^*$ is a principal ultrafilter. Then $\mathscr{N}(x^*) \subseteq \mathfrak{F}$ for some x^* and $\mathfrak{F}^*$ is convergent. Case 2. $\mathfrak{F}^*$ is not a principal ultrafilter. By lemma 6.2, $\mathfrak{F}^* \supseteq \mathfrak{F}$ and hence $\mathfrak{F}^*$ is convergent.

Corollary 7. 2 *If $\mathfrak{F}$ is a filter on X, then $(X, \mathscr{U}(\mathfrak{F}))$ is complete iff $\cap \mathfrak{F} \neq \emptyset$.*

Proof. Suppose $x^* \in \cap \mathfrak{F}$. By theorem 7.1, it suffices to show that $\mathscr{N}(x^*) \subseteq \mathfrak{F}$. If $N \in \mathscr{N}(x^*)$, there exists an $F \in \mathfrak{F}$ such that $(\Delta \cup F \times F)$

$[x^*] \subseteq N$. Then $F \subseteq N$ and $N \in \mathfrak{F}$. Conversely, suppose $(X, \mathscr{U}(\mathfrak{F}))$ is complete. By theorem 7.1, $\mathfrak{F}$ is convergent and hence there exists a point x^* such that $\mathscr{N}(x^*) \subseteq \mathfrak{F}$. We show now that $x^* \in \bigcap \mathfrak{F}$. If $x^* \notin F^* \in \mathfrak{F}$, then $(\Delta \cup F^* \times F^*)[x^*] \subseteq \{x^*\}$ and hence $\{x^*\}$ is open. Then $\{x^*\} \in \mathscr{N}(x^*) \subseteq \mathfrak{F}$ and hence $\{x^*\} \in \mathfrak{F}$. But $\{x^*\} \cap F^* = \emptyset$, a contradiction.

Corollary 7. 3 *If* $(X, \mathscr{U}(\mathfrak{F}))$ *is not separated, then* $(X, \mathscr{U}(\mathfrak{F}))$ *is complete.*

Proof. By theorem 4.3, $\bigcap \mathfrak{F}$ has at least two points and thus $\bigcap \mathfrak{F} \neq \emptyset$. By the preceding theorem, $(X, \mathscr{U}(\mathfrak{F}))$ is complete.

8. **Theercm 8. 1** $(X, \mathscr{U}(\mathfrak{F}))$ *is totally bounded iff* $F \in \mathfrak{F}$ *implies that* $\mathscr{C}F$ *is finite.*

Proof. Sufficiency. Let $U \in \mathscr{U}(\mathfrak{F})$. Then $U \supseteq \Delta \cup F \times F$ for some $F \in \mathfrak{F}$. Let $x_1 \in F$, $\mathscr{C}F = \{x_2, \cdots, x_n\}$. Then $U[x_1, \cdots, x_n] = X$.

Necessity. Suppose $F \in \mathfrak{F}$. Then $\Delta \cup F \times F \in \mathscr{U}(\mathfrak{F})$ and hence there exist x_i such that $(\Delta \cup F \times F)[x_1, \cdots, x_n] = X$. Then $\mathscr{C}F \subseteq (\Delta \cup F \times F)[x_1, \cdots, x_n] \subseteq F \cup \{x_1, \cdots, x_n\}$. Thus $\mathscr{C}F \subseteq \{x_1, \cdots, x_n\}$ and hence $\mathscr{C}F$ is finite.

Theorem 8. 2 $(X, \mathscr{U}(\mathfrak{F}))$ *is compact iff* (1) $\bigcap \mathfrak{F} \neq \emptyset$ *and* (2) $F \in \mathfrak{F}$ *implies that* $\mathscr{C}F$ *is finite.*

The proof follows from theorem 8.1 and corollary 7.2.

9. In this final section, we will be concerned with fg-uniformities for a set X (see § 1).

Theorem 9. 1 *Let* $\mathscr{U} = \bigvee \mathscr{U}_\alpha$ *where* $\mathscr{U}_\alpha$ *is an fg-uniformity for* X *for each* $\alpha \in \Delta$. *Then* $\mathscr{U}$ *is an fg-uniformity for* X.

Proof. For each $\alpha \in \Delta$, there exists a filter $\mathfrak{F}_\alpha$ such that $\mathscr{U}_\alpha = \mathscr{U}(\mathfrak{F}_\alpha)$. Case 1. $F_1 \cap \cdots \cap F_n \neq \emptyset$ for all F_i in $\bigcup \mathfrak{F}_\alpha$. Let $\mathfrak{F} = \bigvee \{\mathfrak{F}_\alpha : \alpha \in \Delta\}$. Then $\mathscr{U}(\mathfrak{F}) = \bigvee \{\mathscr{U}(\mathfrak{F}_\alpha) : \alpha \in \Delta\}$ as the reader can easily verify. Case 2. $F_1^* \cap \cdots \cap F_n^* = \emptyset$ for some F_i^* in $\bigcup \mathfrak{F}_\alpha$. In this case, $\bigvee \mathscr{U}_\alpha$ is discrete since $\Delta = \bigcap \{\Delta \cup F_i^* \times F_i^*\}$. Thus $\mathscr{U}$ is generated by any principal ultrafilter.

Lemma 9. 2 *Let* $\mathfrak{F}$ *be a filter on* X *and let* Y *be a set. Suppose* $f: X \to Y$ *and* $\mathfrak{F}^* = \{F^* : Y \supseteq F^* \supseteq f[F]$ *for some* F *in* $\mathfrak{F}\}$. *Then* f *is uniformly continuous relative to* $\mathscr{U}(\mathfrak{F})$ *and* $\mathscr{U}(\mathfrak{F}^*)$.

Proof. This follows from the identity $\Delta\cup F\times F\subseteq(f\times f)^{-1}(\Delta\cup f[F]\times f[F])$ where Δ is the diagonal in the appropriate space.

Lemma 9. 3 *Let $\mathfrak{F}$ be a filter on X and $\mathscr{V}$ a uniformity for Y. Suppose that $f: X\rightarrow Y$ is uniformly continuous relative to $\mathscr{U}(\mathfrak{F})$ and $\mathscr{V}$. Then $\mathscr{V}\subseteq\mathscr{U}(\mathfrak{F}^*)$ where $\mathfrak{F}^*$ is defined as in lemma* 9.2.

Proof. If $V\in\mathscr{V}$, then $(f\times f)^{-1}[V]\supseteq\Delta\cup F\times F$ for some $F\in\mathfrak{F}$ and it follows that $V\supseteq\Delta\cup f[F]\times f[F]$. Thus $V\in\mathscr{U}(\mathfrak{F}^*)$.

Theorem 9. 4 *Let $f:(X, \mathscr{U})\rightarrow(Y, \mathscr{V})$ be a uniform identification, that is, let f be onto and let $\mathscr{V}$ be the largest uniformity for Y for which f is uniformly continuous relative to $\mathscr{U}$. If $\mathscr{U}$ is an fg-uniformity, then so is $\mathscr{V}$.*

Proof. Apply lemma 9.2 and lemma 9.3.

Theorem 9. 5 *Let $(Y, \mathscr{V})$ be a subspace of $(X, \mathscr{U})$. If $\mathscr{U}$ is an fg-uniformity for X, then $\mathscr{V}$ is an fg-uniformity for Y.*

Proof. Let $\mathscr{U}=\mathscr{U}(\mathfrak{F})$ where $\mathfrak{F}$ is a filter on X. Case 1. $Y\cap F^*=\emptyset$ for some $F^*\in\mathfrak{F}$. Then $\Delta_Y=Y\times Y\cap(\Delta_X\cup F^*\times F^*)\in Y\times Y\cap\mathscr{U}(\mathfrak{F})=Y\times Y\cap\mathscr{U}=\mathscr{V}$. Thus $\Delta_Y\in\mathscr{V}$ and $\mathscr{V}$ is discrete. A discrete uniformity is always fg. Case 2. $Y\cap F\neq\emptyset$ for all F in $\mathfrak{F}$. Then $Y\cap\mathfrak{F}$ is a filter on Y and $\mathscr{V}=\mathscr{U}(Y\cap\mathfrak{F})$ as the reader can check.

REFERENCES

[1] JOHN L. KELLEY: General Topology, D. Van Nostrand Company, Inc., 1955.
[2] WILLIAM J. PERVIN: Foundations of General Topology, Academic Press, 1964.
[3] WOLFGANG J. THRON: Topological Structures, Holt, Rinehart and Winston, 1966.

THE OHIO STATE UNIVERSITY

(*Received March 7, 1969*)