# GROUP RINGS WITH NILPOTENT UNIT GROUPS

Dedicated to Professor Keizo Asano on his 60th birthday

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In their paper [1], J. M. Bateman and D. B. Coleman stated the following: *Let $F$ be a field, and $G$ a finite group.* (a) *Let the group ring $FG$ be semi-simple. Then the unit group of $FG$ is nilpotent if and only if $G$ is abelian.* (b) *Let the characteristic of $F$ be a prime $p$ dividing the order of $G$. Then the unit group of $FG$ is nilpotent if and only if $G$ is a nilpotent group such that the $q$-Sylow subgroup is abelian for every prime $q\neq p$.* Unfortunately, they used there an incorrect lemma, which should be corrected as follows:

**Lemma 1.** *Let $S$ be a ring with* 1, *and $N$ a nilpotent ideal of $S$. If $S/N$ is commutative and $[N, S]=\{[x, y]=xy-yx \mid x\in N,\ y\in S\}$ is contained in $N^2$ then the unit group $S^{\cdot}$ of $S$ is nilpotent. In particular, if $S/N^2$ is commutative then $S^{\cdot}$ is nilpotent.*

*Proof.* We define $(u, v)=u^{-1}v^{-1}uv$ for $u, v\in S^{\cdot}$, and inductively $(u_1, \cdots, u_n)=((u_1, \cdots, u_{n-1}), u_n)$ for $u_1, \cdots, u_n\in S^{\cdot}$. Then, we see by induction that for $n>1$

$(u_1, \cdots, u_n)-1=(u_1, \cdots, u_{n-1})^{-1}u_n^{-1}[(u_1, \cdots, u_{n-1})-1, u_n]\in N^{n-1}$. Since $N$ is nilpotent, it follows that $S^{\cdot}$ is nilpotent.

*Remark.* Let $D=Q+Qi+Qj+Qij$ be the quaternion division algebra over the rational number field $Q$. We consider the ring $S=\{\begin{pmatrix} a & 0 \\ d & a\end{pmatrix} \mid d\in D, a\in C=Q+Qi\}$. Then, $N=\{\begin{pmatrix} 0 & 0 \\ d & 0\end{pmatrix} \mid d\in D\}$ is an ideal of $S$ with $N^2=0$ and $S/N$ is isomorphic to the field $C$. For an arbitrary integer $n$, we have $\begin{pmatrix} 1 & 0 \\ nj & 1\end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix}\begin{pmatrix} 1 & 0 \\ nj & 1\end{pmatrix}^{-1}\begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix}^{-1}=\begin{pmatrix} 1 & 0 \\ nj & 1\end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & i\end{pmatrix}\begin{pmatrix} 1 & 0 \\ -nj & 1\end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & -i\end{pmatrix}$ $=\begin{pmatrix} 1 & 0 \\ 2nj & 1\end{pmatrix}$, whence one will easily see that $S^{\cdot}$ is not nilpotent. This example shows that the assumption $[N, S]\subseteq N^2$ is indispensable in Lemma 1. Next, we shall claim that the converse of Lemma 1 is not true. Evidently the radical $N$ of the ring $S=\{\begin{pmatrix} a & 0 \\ b & c\end{pmatrix} \mid a, b, c\in \mathrm{GF}(2)\}$ coincides with $\{\begin{pmatrix}0&0\\b&0\end{pmatrix}|b\in GF(2)\}$ and $S/N$ is isomorphic to $GF(2)\oplus GF(2)$. Moreover, $S^{\cdot}=\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\1&1\end{pmatrix}\}$ is commutative and $[N,S]\neq 0=N^2$.

Now, we shall prove the following:

**Proposition.** *Let $S$ be a semi-primary ring with 1 such that the radical $R$ is nilpotent and $S^*=S/R^2$ is commutative, and let $G$ be a finite group. If (1) $G$ is commutative or (2) $S/R$ is of prime characteristic $p$ and $G$ is a nilpotent group such that the $q$-Sylow subgroup is commutative for every prime $q\neq p$, then the unit group of the group ring $SG$ is nilpotent.*

*Proof.* We consider the ring homorphism $\lambda$ of $\mathfrak{S}=SG$ onto the group ring $\mathfrak{S}^*=S^*G$ deifined by $\sum_{\sigma\in G}s_\sigma\sigma\longmapsto\sum_{\sigma\in G}s_\sigma^*\sigma$ where $s_\sigma^*$ is the residue class of $s_\sigma\in S$ modulo $R^2$. Evidently, $RG$ is nilpotent and Ker $\lambda$ $=R^2G=(RG)^2$. If $G$ is oommutative then $\mathfrak{S}/(RG)^2$ is isomorphic to the commutative ring $\mathfrak{S}^*$, and hence $\mathfrak{S}^{\cdot}$ is nilpotent by Lemma 1. It remains therefore to prove the case (2). Let $G=H\times P$ where $P$ is a $p$-group and $H$ an abelian group of order prime to $p$. By [3; Corollary 1], the respective radicals $\mathfrak{R}$ and $\mathfrak{R}^*$ of $SP$ and $S^*P$ are $\sum_{\rho\in P}S(\rho-1)$ $+RP$ and $\sum_{\rho\in P}S^*(\rho-1)+(R/R^2)P$. Moreover, noting that $(\mathfrak{R}H)^2$ contains Ker $\lambda$ and $\lambda((\mathfrak{R}H)^2)=(\mathfrak{R}^*H)^2$, we see that $\mathfrak{S}/(\mathfrak{R}H)^2$ is isomorphic to $\mathfrak{S}^*/(\mathfrak{R}^*H)^2$. As $H$ is contained in the center of $\mathfrak{S}^*$ and $[\sigma,\tau]=[\sigma-1,\tau-1]\in(\mathfrak{R}^*H)^2$ for every $\sigma,\tau\in P$, it is easy to see that ($\mathfrak{S}^*/(\mathfrak{R}^*H)^2$ and hence) $\mathfrak{S}/(\mathfrak{R}H)^2$ is commutative. As was noted in the proof of [3; Corollary 1], $\mathfrak{R}^k$ is contained in $RP$ for some $k$, which implies that $\mathfrak{R}H$ is nilpotent. Hence, again by Lemma 1, $\mathfrak{S}^{\cdot}$ is nilpotent.

As is well-known, the unit group of the complete $n\times n$ matrix ring $D_n$ over a division ring $D$ is not nilpotent for $n>1$. Moreover, it is known that the unit group of a division ring $D$ is not solvable if $D$ is not commutative ([2] or [4]). Accordingly, we readily obtain

**Lemma 2.** *If the unit group of an artinian semi-simple ring $S$ is nilpotent then $S$ is commutative.*

Combining the proposition with Lemma 2, we can generalize somewhat the statement cited at the opening of this note.

**Theorem.** *Let $S$ be an artinian semi-simple ring, and $G$ a finite group. Then, the unit group of the group ring $SG$ is nilpotent if and* *only if $S$ is commutative and either* (1) *$G$ is abelian or* (2) *$S$ is of prime characteristic $p$ and $G$ is a nilpotent group such that the $q$-Sylow subgroup is commutative for every prime $q \neq p$.*

*Proof.* By the validity of our proposition, it suffices to prove the only if part. If $S$ is simple and the characteristic of $S$ does not divide the order of $G$ then, as is well-known, $SG$ is artinian semisimple. Hence, $S$ and $G$ must be commutative by Lemma 2. Next, if $S$ is a simple ring of prime characteristic $p$ dividing the order of $G$ then by the fact noted just above $S$ and every $q$-Sylow subgroup of $G$ are commutative ($q \neq p$). Now, combining those above, we can readily complete our proof.

Although the converse of our proposition is not valid, we obtain the following:

**Corollary.** *Let $S$ be a semi-primary ring with* 1, *and $G$ a finite group. If the unit group of $SG$ is nilpotent then the residue class ring $\bar{S}$ of $S$ modulo its radical $R$ is commutative and either* (1) *$G$ is commutative or* (2) *$\bar{S}$ is of prime characteristic $p$ and $G$ is a nilpotent group such that the $q$-Sylow subgroup is commutative for each prime $q \neq p$.*

*Proof.* We consider the ring homomorphism $\mu$ of $SG$ onto $\bar{S}G$ defined by $\sum_{\sigma \in G} s_\sigma \sigma \longmapsto \sum_{\sigma \in G} \bar{s}_\sigma \sigma$ where $\bar{s}_\sigma$ is the residue class of $s_\sigma$ modulo $R$. As is well known, Ker $\mu = RG$ is contained in the radical of $SG$, and so the unit group of $\bar{S}G$ is nilpotent. Hence, the corollary is evident by our theorem.

## REFERENCES


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**Added in proof.** After the submission of this manuscript, the writers have learned that K. Eldridge has submitted a short paper that correct the error in [1]. Also P. B. Bhattacharya and S. K. Jain [Notices of Amer. Math. Soc. 16 (1969), 562] have presented a counterexample to the lemma of [1], provided another proof for the theorem of [1], and shown that if $S$ is an artinian ring with 1 and $G$ is a finite group such that the unit group of $SG$ is nilpotent then $SG$ satisfies a polynomial identity $(xy-yx)^n=0$. Indeed, the last is an easy consequence of our theorem.