ON FINITE DIMENSIONALITIES OF RING EXTENSIONS FOR PRIMITIVE RINGS WITH NON-ZERO SOCLES

Dedicated to Professor Keizo Asano on his 60th birthday

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Let A be an arbitrary ring and B a subring of A. As in [2], we shall say that A has a right dimensionality over B if B is a primitive ring with a non-zero socle, that is, a ring having faithful minimal right ideals and $I'A$ is a faithful, homogeneous completely reducible B-module for some miniaml right ideal I' of B. In this case the right dimensionality $[A:B]_R$ is defined to be the cardinal number of irreducible direct summands of B-module $I'A$.

In this paper, we shall make some remarks on the results of [1, Th. 2 and its corollary] and [2, VI. §§6, 7] which are the study of dimensionalities for primitive rings with non-zero socles, where this paper depends heavily on [2].

Throughout the present paper, when M is a right A-module (resp. a left A-module), we place ourselves in the situation described by the symbol M_A (resp. ${}_AM$). Moreover, when A has a right A-module M (resp. a left A-module M), we place ourselves in the situation described by the symbol ${}_{(M)}A$ (resp. $A_{(M)}$). For a right A-module M, we have $M_{A,L}$ where L is the ring of all the A-endomorphisms of M; and by ${}_{(M)}\bar{A}$ we denote the ring of all the L-endomorphisms of M. If M is a faithful right A-module then ${}_{(M)}\bar{A}$ contains A, where for every $a\in A$, $a\in{}_{(M)}\bar{A}$ means the mapping $x\rightarrow xa$ $(x\in M)$. If $A_1\supset B_1$ and $A_2\supset B_2$ are rings and there exists a ring isomorphism F of A_1 onto A_2 such that B_1 is isomorphic to B_2 under the mapping F then we write $A_1/B_1\cong A_2/B_2\ (F)$; if, in particular, $B_1=B_2=B$ and $xF=x$ $(x\in B)$ then we write $A_1/B\cong A_2/B\ (F/I)$.

We note first the following lemma which is well known.

Lemma 1. *Let M be a faithful right A-module and $M=\sum_{i\in I}N_i$ a direct sum of A-submodules where all the N_i are mutually A-isomorphic. If N is one of the N_i then ${}_{(M)}\bar{A}/A\cong{}_{(N)}\bar{A}/A\ (F/I)$ where for $f\in{}_{(M)}\bar{A}$, fF is the restriction of f to N.*

Throughout the rest of this note, we shall understand by a primitive ring a right primitive ring, that is, a ring which has a faithful irreducible right module. If A is a primitive ring having minimal right ideals then every minimal right ideal of A is a faithful right A-module, which is isomorphic to every faithful irreducible right A-module ([2, Prop. III. 5. 2]); and the socle of A is the sum of the minimal right ideals of A ([2, pp. 63, 64]). If A is a primitive ring having no minimal right ideals then we say that the socle of A is zero ([2, p. 63]). If B is a subring of A which is primitive and has a non-zero socle then, for any two minimal right ideals I_1, I_2 of B, I_1A is A-isomorphic to I_2A ([2, Prop. III. 7. 4, Prop. III. 9. 1]).

By the above remarks and Lemma 1, we have the following

Corollary 1. *Let A be a primitive ring with a non-zero socle. Let M and M' be right A-modules which are faithful, homogeneous completely reducible. Then, there exists a ring isomorphism F of ${}_{(M)}\bar{A}$ onto ${}_{(M')}\bar{A}$ such that*

(a) ${}_{(M)}\bar{A}/A \cong {}_{(M')}\bar{A}/A \quad (F/I)$, *and*

if B is a subring of A then

(b) ${}_{(M)}\bar{A}/{}_{(M)}\bar{B} \cong {}_{(M')}\bar{A}/{}_{(M')}\bar{B} \quad (F)$.

In case Coro. 1 (a), we write $\bar{A} = {}_{(M)}\bar{A}$. This is a homogeneous distinguished ring of endomorphisms in the sense of [2, Def. VI. 3. 1]. In case Coro. 1 (b), if, in addition, M is homogeneous completely reducible as B-module, by $\bar{A} \supset_{(*)} \bar{B}$ we denote the situation ${}_{(M)}\bar{A} \supset {}_{(M)}\bar{B}$.

For primitive rings, we have Dieudonné, two notions: Height and index ([1], [2]) which are as follows: Let $A \supset B$ be primitive rings.

(a) If A has a non-zero socle and some minimal right ideal I of A is a faithful, homogeneous completely reducible B-module, then we define the (right) height of A over B to be the cardinal number of irreducible direct summands of B-module I.

(b) If B has a non-zero socle and for some minimal right ideal I' of B, $I'A$ is a completely reducible A-module, then we define the (right) index of A over B to be the cardinal number of irreducible direct summands of A-module $I'A$.

In case (b), $I'A$ contains a minimal right ideal of A, and whence, the soele of A is non-zero. Thus, the above definitions and [2, Prop. VI. 6. 1] imply the following

Lemma 2. *Let $A \supset B$ be primitive rings.*

(a) *Let the socle of A be non-zero. If there exists the height of A over B then $\bar{A} \supset_{(*)} \bar{B}$, and conversely.*

(b) *Let the socle S' of B be non-zero. If there exists the index of A over B then A has a non-zero socle S such that $S \supset S'$, and conversely.*

Remark. Let $A \supset B$ be primitive rings with non-zero socles S, S' respectively. If there exists the height of A over B then the situation $\bar{A} \supset_{(*)} \bar{B}$ (resp. $\bar{A} =_{(*)} \bar{B}$) may be written as $\bar{A} \supset \bar{B}$ (resp. $\bar{A} = \bar{B}$). A right A-module M is faithful, homogeneous completely reducible if and only if $MS = M$ ([2, Th. IV. 14. 1]). Hence, there exists the height of A over B if and only if $IS' = I$ for every minimal right ideal I of A. If there exists the index of A over B then the index is finite ([2, Prop. VI. 6. 1]).

The following corollary will be easily seen.

Corollary 2. *Let $A \supset B$ be primitive rings with non-zero socles S, S' respectively, and let M be a right A-module.*

(a) *If there exists the height of A over B and M_A is faithful, homogeneous completely reducible then the ring extension ${}_{(M)}B \to {}_{(M)}\bar{A}$ coincides with the composed ring extensions $B \to \bar{B} \to \bar{A}$, $B \to A \to \bar{A}$.*

(b) *When there exist the height and the index of A over B, M_A is faithful, homogeneous completely reducible if and only if so is M_B.*

The following proposition is a slight variation of the result of [2, Prop. VI. 6. 3].

Proposition 1. *Let A be a primitive ring, and B a subring of A which is primitive and has a non-zero socle S'. If $[A:B]_R$ exists and is finite then A has a non-zero socle S such that $S \supset S'$.*

Proof. Let I' be a minimal right ideal of B. Then $I'A$ has a finite composition series as B-module. Hence $I'A$ has a finite composition series as A-module. Therefore $I'A$ contains a minimal right ideal of A. Thus A has a non-zero socle. Then we have $S \supset S'$ by [2, Prop. VI. 6. 3].

The following corollary is a direct consequence of Prop. 1 and [2, Prop. VI. 6. 1].

Corollary 3. *Let A be a primitive ring, and B a subring of A which is primitive and has a non-zero socle. Then, $[A:B]_R$ exists and is finite if and only if the index of A over B exists and the height of A over B exists and is finite; and when this is so $[A:B]_R$ is the product of the height and the index.*

The following proposition contains the result of [2, Coro. VI. 6. 1].

Proposition 2. *Let A be a primitive ring, and B a subring of A which is primitive and has a non-zero socle. Let T be an intermediate ring of A/B which is primitive.*

(a) *If $[A:B]_R$ exists and is finite then the socle of T is non-zero, $[A:T]_R$ and $[A:B]_R$ exist, and $[A:B]_R=[A:T]_R[T:B]_R$.*

(b) *Let the socle of T be non-zero. If $[A:T]_R$ and $[T:B]_R$ exist and are finite then $[A:B]_R=[A:T]_R[T:B]_R$.*

Proof. Our assertion (b) is a direct consequence of Prop. 1 and [2, Coro. VI. 6. 1]. The proof of (a) is as follows. Since $[A:B]_R$ is finite, by Prop. 1, the socle S of A is non-zero and contains the socle S' of B. If I' is a minimal right ideal of B then $\{0\}\neq I'T\subset I'A$, and which are faithful, homogeneous completely reducible B-modules. Hence $[T:B]_R$ exists and is finite. By Prop. 1, the socle S^* of T is non-zero and contains S'. Moreover we have $I'T\subset S^*$ and $I'A\subset S$. Since $S^*\cap S(\supset I'T\neq\{0\})$ is an ideal of S^* and S^* is a simple ring, it follows that $S^*\subset S$. Thus the index of A over T exists by Lemma 2. Let I be a minimal right ideal of A. Then $IS'=I$, and so, $IS^*=I$. Hence the height of A over T exists by the remark of Lemma 2. Moreover the height is finite. Therefore, it follows from Coro. 3 that $[A:T]_R$ exists and is finite. Thus, by (b), we have our assertion (a).

Now, we shall prove the following proposition which is useful in the rest of our study.

Proposition 3. *Let A be a ring, and B a subring of A which is primitive and has a non-zero socle. Then the following conditions are equivalent.*

(a) *A is primitive, $[A:B]_R$ exists and equals to* 1.

(b) *$B\subset A\subset\bar{B}$.*

(c) *$\bar{A}=\bar{B}$ (in the sense of the remark of Lemma* 2).

Particularly we have $[\bar{B}:B]_R=1$.

Proof. Let I' be a minimal right ideal of B. Then $I'=e'B$, where e' is a minimal idempotent of B ([2, Prop. III. 10. 1]). Moreover Hom $(e'B_B, e'B_B)$ is a division ring, and which is the ring of all the endomorphisms $x \to ax$ $(x \in e'B)$, $a \in e'Be'$ ([2, Prop. III. 7. 3]). Now, we shall give a cyclic proof of our proposition in the order (a)⇒(c)⇒(b)⇒(a). Assume (a). Then $e'A=(e'B)A=e'B$, and this is a minimal right ideal of A. Since A is primitive, $e'A$ is a faithful irreducible right A-module. Since $e'Ae'=e'Be'$, we have Hom $(e'A_A, e'A_A)=$Hom $(e'B_B, e'B_B)$. Hence we obtain $\bar{A}=\bar{B}$. This proves that (a) implies (c). Next (c)⇒(b) is obvious. Assume (b). Then $\bar{B}/B={}_{(e'B)}\bar{B}/B$ (Coro. 1), and A has a faithful irreducible right A-module $e'B$. Hence A is a primitive ring. Let S' be the socle of B. Since $e'B=e'S'$ and is a faithful irreducible right S'-module, we have $\overline{S'}=\bar{B}\supset A\supset B\supset S'$. Hence, by Prop. 2 (a), it suffices to prove that $[\overline{S'}:S']_R=1$. If $f\in S'$ and $g\in\overline{S'}$ then $e'S'f$ is a finitely generated left $e'S'e'$-module ([2, p. 75, Structure theorem for primitive rings with non-zero socles]), and whence, by the density theorem for irreducible modules ([2, p. 31]), the restriction of g to $e'S'f$ coincides with the restriction of some $h\in S'$ to $e'S'f$, which implies $fg=fh\in S'$. Hence $S'\overline{S'}\subset S'$, that is, $S'\overline{S'}=S'$. Then we have ${}_{(e'S')}e'S'\overline{S'}={}_{(e'S')}e'S'$. This implies $[\overline{S'}:S']_R=1$.

Proposition 4. *Let $A\supset A'\supset B$ be primitive rings and let the socle of B be non-zero. Assume that $[A:B]_R$ exists and is finite. Then $[A:B]_R=[A':B]_R$ if and only if $\bar{A}=\bar{A}'$.*

Proof. By Prop. 2, $[A:A']_R$ and $[A':B]_R$ exist and $[A:B]_R=[A:A']_R[A':B]_R$. Hence $[A:B]_R=[A':B]_R$ if and only if $[A:A']_R=1$, and this is equivalent to that $\bar{A}=\bar{A}'$ (Prop. 3).

Proposition 5. *Let A be a primitive ring, and B a subring which is primitive and has a non-zero socle. If $[A:B]_R$ exists and is finite then $[\bar{A}:\bar{B}]_R=[A:B]_R$.*

Proof. Since $\bar{A}\supset A\supset B$ and $\bar{A}\supset\bar{B}\supset B$ (Coro. 2), it follows from Prop. 2 and Prop. 3 that $[\bar{A}:A]_R[A:B]_R=[\bar{A}:B]_R=[\bar{A}:\bar{B}]_R[\bar{B}:B]_R$. Noting that $[\bar{A}:A]_R=1$ and $[\bar{B}:B]_R=1$ (Prop. 3), we obtain $[A:B]_R=[\bar{A}:\bar{B}]_R$.

The following proposition contains the results of [1, Coro. of Th. 2] and [2, Th. VI. 7. 1].

Proposition 6. *Let* $A \supset B$ *be primitive rings such that* B *has a non-zero socle and* $[A:B]_R$ *exists and is finite. Let* M *be a right* A*-module. Then,* M_A *is faithful, homogeneous completely reducible if and only if so is* M_B. *In this case, if* $L = Hom(M_A, M_A)$ *and* $L' = Hom(M_B, M_B)$ *then* $[L':L]_R = [A:B]_R$.

Proof. By Coro. 2 and Coro. 3, M_A is faithful, homogeneous completely reducible if and only if so is M_B. Assume the conditions. Then $\bar{A} \supset \bar{B}$ are homogeneous distinguished rings of endomorphisms of M ([1, Def. VI. 3. 1]). Therefore, by [2, Th. VI. 7. 1], we have $[L':L]_R = [\bar{A}:\bar{B}]_R$. Since $[\bar{A}:\bar{B}]_R = [A:B]_R$ (Prop. 5), it follows that $[L':L]_R = [A:B]_R$.

Remark. Let A, B, M, L, and L' be as in the preceding proposition, and let M_A $(=M)$ be faithful, homogeneous completely reducible. Then the index of A over B equals to the index of $\bar{A}$ over $\bar{B}$ and the height of A over B equals to the height of $\bar{A}$ over $\bar{B}$. Hence, by [1, Th. 2] (or the proof of [2, Th. VI. 7. 1]), the index of A over B equals to the height of L' over L and the height of A over B equals to the index of L' over L. This is a generalization of [1, Th. 2].

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