A CHARACTERIZATION OF QUATERNION ALGEBRAS

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In the present note, we shall prove the following theorem with a quite elementary proof.

Theorem. *Let $A(\ni 1)$ be a central simple (Artinian) algebra over C whose characteristic is different from 2. If $A' = \{x \in A \setminus C$ (complement of C in A); $x^2 \in C\} \cup \{0\}$ forms a non-zero additive group then A is a quaternion algebra (and conversely).*

Proof. We claim first that $A' \oplus C$ is a subring of A. In fact, if x, y are in A' then $c(x, y) = xy + yx = (x+y)^2 - x^2 - y^2 \in C$, and then $(xy)^2 = x(c(x, y) - xy)y = c(x, y) \cdot xy - x^2y^2$. Hence, $(xy - c(x, y)/2)^2 \in C$, whence it follows $xy \in A' \oplus C$. As a direct consequence of this fact, we have $(xz)y + y(xz) \in C \oplus Cy$ for $x, y, z \in A'$. Now, let $A = \sum_1^n De_{ij}$, where $E = \{e_{ij}\}$ is a system of matrix units and $D = V_A(E)$ (centralizer of E in A) a division ring. We shall distinguish here between two cases.

Case I. $n = 1$: To be easily seen, $A' \oplus C$ is algebraic over C, and hence a division ring. Evidently, there exist some $x \in A'$ and $u_0 \in A$ such that $xu_0 \neq u_0x$. If u is an arbitrary element of A such that $x \neq uxu^{-1} = x_1$ $(\in A')$ then $x \neq (u+1)x(u+1)^{-1} = x_2 (\in A')$. We have then $x - x_2 = (x_2 - x_1)u$. whence it follows $u = (x - x_2)(x_2 - x_1)^{-1} \in A' \oplus C$. On the other hand, if v is an arbitrary element of $V_A(x)$ then $x \neq (u_0 + v)x(u_0 + v)^{-1}$. Accordingly, by the above, $v = (u_0 + v) - u_0 \in A' \oplus C$. We have seen thus $A' \oplus C = A$. [1] Evidently, there exists then an element $y \in A' \setminus C[x]$ and there holds $C[x, y] = C \oplus Cx \oplus Cy \oplus Cxy$. Now, suppose $A \neq C[x, y]$, and take an arbitrary element z from $A' \setminus C[x, y]$. Then, $C[x, y, z] = C \oplus Cx \oplus Cy \oplus Cz \oplus Cxy \oplus Cyz \oplus Czx \oplus Cxyz$. However, as $(xz)y + y(xz) \in C \oplus Cy$ by the remark stated at the opening of this proof, $xyz = x(c(y, z) - zy) = c(y, z) \cdot x - (xz)y$ and $xyz = (c(x, y) - yx)z = c(x, y) \cdot z - y(xz)$ yield a contradibtion $xyz \in C \oplus$

1) Then, as was noted in [1; p. 578], A satisfies the polynomial identity $(xy - yx)^2x - x(xy - yx)^2 = 0$, and so A is a quaternion division algebra by [2; Th. 6.2]. Another elementary proof is given in [3].

$Cx \oplus Cy \oplus Cz$. Hence, $A = C[x, y]$ and $[A : C] = 4$.

Case II. $n > 1$: We have $n = 2$. In fact, if $n > 2$ then $(e_{12} + e_{23})^2 = e_{13}$ does not belong to C. Next, suppose $D \neq C$, and choose two elements $a, b \in D$ such that $ab \neq ba$. We have then $(ae_{12} + be_{21})^2 = abe_{11} + bae_{22} \notin C$. This contradiction proves our assertion $A = \sum_1^2 Ce_{ij}$.

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