A REMARK ON LYAPUNOV-HALMOS-BLACKWELL'S CONVEXITY THEOREM

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1. In this note, we shall give another simple proof of the following theorem due to Lyapunov, Halmos and Blackwell: *Let* $\mu_1, \mu_2, \cdots, \mu_n$ *be non-atomic completely additive measures on a* σ*-complete Boolean lattice* $\mathfrak{B}$ *and their total variations be finite. Then the set of all points* $(\mu_1(B), \mu_2(B), \cdots, \mu_n(B))$ *for* $B \in \mathfrak{B}$ *is convex in the n-dimensional space.* We use only elementary tool of measure theory for proving this theorem. As for the closedness theorem, we can not improve the proof of Blackwell.

At the final part of this note, we shall show that this theorem can not be generalized even in infinite-dimensional nuclear spaces. But this theorem is true in strictly inductive limit of Euclidean spaces which is infinite dimensional.

2. Let μ be a completely additive measure on σ-complete Boolean lattice $\mathfrak{B}$, whose total variation is finite. An element $B \in \mathfrak{B}$ is an *atom* for μ, if $\mu(B) \neq 0$ and $\mu(X) = \mu(B)$ or $=0$ for all $X \subset B$, $X \in \mathfrak{B}$. μ is called *non-atomic* if there is no atom for μ. It is easily seen that $\mu^+ (= \mu \cup 0)$ and $\mu^- (= (-\mu) \cup 0)$ are non-atomic if μ is non-atomic.

Lemma 1. *The following conditions are equivalent*:

(1) μ *is non-atomic.*

(2) *For every* $\varepsilon > 0$ *and* $F \in \mathfrak{B}$ *with* $\mu(F) \neq 0$, *there exists* $B \subset F$ *with* $B \in \mathfrak{B}$ and $0 < |\mu(B)| \leqq \varepsilon$.

(3) *For every real number* $1 > \lambda > 0$ *and* $F \in \mathfrak{B}$, *there exists* $B \in \mathfrak{B}$ *with* $B \subset F$ *and* $\mu(B) = \lambda \mu(F)$.

Proof. (1)$\Rightarrow$(2) If μ is non-atomic and $\mu(F) \neq 0$ where $F \in \mathfrak{B}$, then we can find $X, Y \in \mathfrak{B}$, $X, Y \subset F$ such that $\mu^+(X) \leqq \varepsilon/2$, $\mu^+(Y) = 0$ and $\mu^-(Y) \leqq \varepsilon/2$, $\mu^-(X) = 0$ with $X \cap Y = \phi$ (minimum of $\mathfrak{B}$) and either $\mu^+(X) \neq 0$ or $\mu^-(Y) \neq 0$. If $\mu^+(X) \neq \mu^-(Y)$, then $B = X \cup Y$ has a desired property. If $\mu^+(X) = \mu^-(Y)$, then we can replace X or Y so that $\mu^+(X) \neq \mu^-(Y)$ by the non-atomicity of μ^+ and μ^-.

(2)$\Rightarrow$(3) This can be proved by Zorn's lemma.

(3)$\Rightarrow$(1) It is clear.

Lemma 2. *If μ_1 and μ_2 are non-atomic on $\mathfrak{B}$, then $\mu_1+\mu_2$ is non-atomic on $\mathfrak{B}$.*

Proof. Since maximum $\mathbf{1}$ of $\mathfrak{B}$ can be decomposed into $E_1\cup E_2=\mathbf{1}$, where $E_i(i=1, 2)$ are mutually disjoint and μ_1 is positive or negative on E_i, we can suppose that $\mu_1\geqq 0$ or $\mu_1\leqq 0$.

Now, suppose that we find an atom $F\in\mathfrak{B}$ for $\mu_1+\mu_2$. For every $n=1, 2, \cdots$, there exists $F_n\in\mathfrak{B}$ with $(1/2^n)\mu_1(F)=\mu_1(F_n)$; $F\supset F_1\supset F_2\cdots\supset F_n\supset\cdots$ and $(\mu_1+\mu_2)(F_n)=(\mu_1+\mu_2)(F)\neq 0(n=1, 2, \cdots)$. For $G=\bigcap_{n=1}^{\infty}F_n$, we have $\mu_1(G)=0$. Hence, $\mu_2(G)=(\mu_1+\mu_2)(G)=\lim_{n\to\infty}(\mu_1+\mu_2)(F_n)=(\mu_1+\mu_2)(F)\neq 0$ and $\mu_2(G_0)=(\mu_1+\mu_2)(G_0)=(\mu_1+\mu_2)(F)$ or $=0$ for every $G_0\subset G$, $G_0\in\mathfrak{B}$. But, this contradicts to the non-atomicity of μ_2.

Lemma 3. *If μ_1 and μ_2 are measures with finite variation on $\mathfrak{B}$ and $\mu_1(\mathbf{1})=\mu_2(\mathbf{1})$, then for every $1>\lambda>0$, there exists $F\in\mathfrak{B}$ with $\mu_1(F)=\mu_2(F)=\lambda\mu_1(\mathbf{1})$.*

Proof. Since $(\mu_1^+ +\mu_2^-)(\mathbf{1})=(\mu_1^- +\mu_2^+)(\mathbf{1})$, we may suppose that $\mu_1\geqq 0$ and $\mu_2\geqq 0$. It suffices to prove that μ_1 and μ_2 are non-atomic on some non trivial Boolean sublattice $\mathfrak{B}_1\subset\{G; \mu_1(G)=\mu_2(G), G\in\mathfrak{B}\}$.

We can decompose $\mathbf{1}$ into $\mathbf{1}=E_1\cup E_2$, $E_1\cap E_2=\phi$; $E_1, E_2\in\mathfrak{B}$; $\mu_1-\mu_2$ is positive on E_1 and $\mu_2-\mu_1$ is positive on E_2 with $(\mu_1-\mu_2)^+(E_2)=(\mu_1-\mu_2)^-(E_1)=0$. If $(\mu_1-\mu_2)^+=0$ or $(\mu_2-\mu_1)^+=(\mu_1-\mu_2)^-=0$, then $\mu_1=\mu_2$ because of $\mu_1(\mathbf{1})=\mu_2(\mathbf{1})$. If $(\mu_1-\mu_2)^+\neq 0$ and $(\mu_2-\mu_1)^-\neq 0$, for $\varepsilon>0$ we can get $F_1\subset E_1$, $F_2\subset E_2$, F_1, $F_2\in\mathfrak{B}$ with $0<(\mu_1-\mu_2)^+(F_1)\leqq\mu_1(F_1)\leqq\varepsilon/2$, $0<(\mu_2-\mu_1)^+(F_2)\leqq\mu_2(F_2)\leqq\varepsilon/2$, by virtue of the non-atomicity of μ_1 and μ_2.

By virtue of Lemma 2, if $0<(\mu_1-\mu_2)^+(F_1)=\delta<\delta'=(\mu_2-\mu_1)^+(F_2)$, then F_2 may be changed so that $(\mu_2-\mu_1)^+(F_2)=\delta$ and both $\mu_i(F_i)$ $(i=1, 2)$ are not identically 0 at the same time.
Hence, we have

$$(\mu_1-\mu_2)(F_1\cup F_2)=(\mu_1-\mu_2)^+(F_1\cup F_2)-(\mu_2-\mu_1)^+(F_1\cup F_2)$$
$$=(\mu_1-\mu_2)^+(F_1)-(\mu_2-\mu_1)^+(F_2)=0$$

and so

$$0<\mu_1(F_1\cup F_2)=\mu_2(F_1\cup F_2)\leqq\varepsilon/2+\varepsilon/2.$$

Hence, we can construct a non trivial Boolean latlice $\mathfrak{B}_1\subset\{G; \mu_1(G)=\mu_2(G), G\in\mathfrak{B}\}$ such that μ_1 and μ_2 are non-atomic on $\mathfrak{B}_1$ and $\mathfrak{B}_1$ contains all $G\in\mathfrak{B}$ with $\mu_1(G)=\mu_2(G)=0$.

Lemma 4. *If $\mu_i (i=1, 2, \cdots, n)$ are non-atomic and positive or negative on $\mathfrak{B}$ and $\mu_i(1)$ coincide with each other, then μ_n is non-atomic with respect to some Boolean sublattice $\mathfrak{B}_{n-1}$ which is contained in $\{B; \mu_1(B) = \cdots = \mu_n(B),\ B\in\mathfrak{B}\}$.*

Proof. By induction, we suppose that μ_1 is non-atomic on some Boolean sublattice $\mathfrak{B}_{n-2}\subset\{B; \mu_1(B)= \cdots =\mu_{n-1}(B),\ B\in\mathfrak{B}\}$ such that $\mathfrak{B}_{n-2}$ contains all $B\in\mathfrak{B}$ with $\mu_1(B)= \cdots =\mu_{n-1}(B)=0$. If μ_n is non-atomic on $\mathfrak{B}_{n-2}$, then μ_n is non-atomic on some $\mathfrak{B}_{n-1}$ by Lemma 3. Suppose now that $E\in\mathfrak{B}_{n-2}$ is an atom for μ_n with respect to $\mathfrak{B}_{n-2}$. For every positive integer m, $\mu_1(F_m)=(1/2^m)\mu_1(E)$, $\mu_n(F_m)=\mu_n(E)\neq 0$, $F_m\in\mathfrak{B}_{n-2}$ with $E\supset F_1\supset F_2\supset\cdots\supset F_m\supset\cdots$. For $G=\bigcap_{m=1}^{\infty} F_m$, $\mu_1(G)=0$ and G is an atom for μ_n. For every $H\subset G$, $H\in\mathfrak{B}$, $\mu_1(H)=\mu_2(H)=\cdots=\mu_{n-1}(H)=0$ i. e. $H\in\mathfrak{B}_{n-2}$. But μ_n is non-atomic on $\{G_0; G_0\subset G,\ G_0\in\mathfrak{B}\}$, so we have a contradiction.

Proof of Theorem.

We can decompose 1 into $1=\bigcup_{i=1}^{s} E_i$, where $\mu_1, \cdots, \mu_n$ are positive or negative on E_i. By Lemma 4, we can easily obtain the conclusion. (one may remark that in the proof of our theorem, Lemma 3 is essential.)

3. Let R be a locally convex space and $\mathfrak{B}$ be a Boolean lattice. We call a mapping μ from $\mathfrak{B}$ to R *R-valued measure* if $\mu(\bigcup_{i=1}^{\infty} B_i)=\sum_{i=1}^{\infty}\mu(B_i)$ (by the topology of R) for mutually disjoint family B_i $(i=1, 2, \cdots)$ of $\mathfrak{B}$.

Let R be strictly inductive limit of Euclidean spaces i. e. $R=\bigcup_{n=1}^{\infty} R_n$ where R_n is n-dimensional with $R_n\subset R_{n+1}$. The range of R-valued measure on non-atomic Boolean lattice $\mathfrak{B}$ is closed and convex because the set $\{\mu(B)\cap R_n; B\in\mathfrak{B}\}$ is closed and convex for all $n=1, 2, \cdots$.

Proposition. *Let R be a nuclear vector lattice which is not contained in inductive limit of Euclidean spaces. Then, there exists a R-valued measure μ on Borel sets of interval* $[0, 1]$ *such that its range is not convex.* (For the definition of nuclear vector lattice, see [3]).

Proof. For every $n=0, 1, 2, \cdots$, we shall consider a family of subsets $\{U\}$ of the interval $[0, 1]$ formed by unions of the subsets: $\left[0, \frac{1}{2^n}\right)$, $\left[\frac{1}{2^n}, \frac{2}{2^n}\right)$, $\cdots$, $\left[\frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}\right)$, $\left[\frac{2^n-1}{2^n}, 1\right]$ such that $m(U)=\frac{1}{2}$, where m is Lebesque measure on $[0, 1]$.

Arranging these subsets for all $n=1, 2, \cdots$, we shall assign numbers: $U_1, U_2, \cdots$.

Note that the linear subspace of $L_1[0, 1]$ generated by the characteristic functions of U_n (denoted by χ_{U_n}) is dense in $L_1[0, 1]$.

If $m(E)=\frac{1}{2}$ for a Borel set E, then $m(E^c)=\frac{1}{2}$ and $m(E\cap U_n)=m(E^c\cap U_n)$ if $m(E\cap U_n)=\frac{1}{4}$.

By [3], R is considered as generalized sequence space. Since R is not inductive limit of Enclidean space by assumption, we can find an element $x\in R$ whose co-ordinates are not 0 at least for infinite countable index. Without loss of generality, we can assume that $x=(a_0, a_1, a_2, \cdots, a_n, \cdots)$ where $a_i \neq 0 (i=0, 1, 2, \cdots)$.
Now, we shall define R-valued measure μ on a Borel field $\mathfrak{B}$ of $[0, 1]$ as follows:

$$\mu(E)=(a_0 m(E),\ a_1\, m(E\cap U_1), \cdots, a_n m(E\cap U_n), \cdots)$$

where $E\in\mathfrak{B}$ and m is Lebesque measure on $\mathfrak{B}$.

It is easy to see that μ is R-valued measure on $\mathfrak{B}$. If the range of μ is convex, then we must have a Borel set $E\in\mathfrak{B}$ such that

$$\mu(E)=\left(\frac{a_0}{2}, \frac{a_1}{4}, \frac{a_2}{4}, \cdots, \frac{a_n}{4}, \cdots\right),$$

since $\mu([0, 1])=(a_0, \frac{a_1}{2}, \cdots\ \frac{a_n}{2}, \cdots)$ and $\mu(\phi)=(0, 0, \cdots 0, \cdots)$.

Hence, $m(E)=\frac{1}{2}$ and $m(E\cap U_n)=\frac{1}{4}=m(E^c\cap U_n)$.

Since

$$\int_0^1 \chi_{U_n}(\chi_E-\chi_{E^c})dm=\int_{U_n}(\chi_E-\chi_{E^c})dm=m(E\cap U_n)-m(E^c\cap U_n)=0$$

and the linear subspace generated by $\{\chi_{U_n}\}$ is dense in $L_1[0, 1]$, we have $\chi_E-\chi_{E^c}=0$ considering as an element of $L_\infty[0, 1]$. This is a contradiction i. e. the range of $\mathfrak{B}$ by μ is not convex.

Since every nuclear F-space with bases is considered as nuclear vector lattice, this proposition is true if R is infinite dimensional nuclear F-space with bases. (c. f. [5])

It is noticed that the range of $\mathfrak{B}$ by μ is convex if the range is weakly closed.

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