A NOTE ON NORMAL BASIS ELEMENTS

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Throughout the present note, A will represent a ring with 1, and B a subring of A containing 1. Recently, in his paper [4], K. Motose obtained the following :

Proposition. *Let A be a primary ring with (Jacobson) radical $\mathfrak{N}$, and G a group of finite order. If $G' \neq \{1\}$ is a normal subgroup of G, $\bar{G} = G/G'$ and $\bar{A} = A/\mathfrak{N}$, then the following conditions are equivalent:*

(1) *$\sum_{\sigma\in G} \sigma a_\sigma$ is a unit of the group ring GA whenever $\sum_{\sigma\in G} \bar{\sigma} \bar{a}_\sigma$ is a unit of $\bar{G}\bar{A}$.*

(2) *The characteristic of $\bar{A}$ is a prime p and the order of G' is a power of p.*

In below, as an application of the proposition, we shall present a slight generalization of [6; Th. 4.2], which is stated as follows :[1]

Theorem. *Let A/B be G-Galois, and $G' \neq \{1\}$ a normal subgroup of G with $B' = J(G')$. If B is a strongly primary ring with (Jacobson) radical $\mathfrak{N}(B)$, $\bar{B} = B/\mathfrak{N}(B)$ and $\bar{G} = G/G'$, then the following conditions are equivalent :*

(1) *a is a right G-n. b. e. of A/B whenver $T_{G'}(a) = \sum_{\sigma'\in G'} a\sigma'$ is a right $\bar{G}$-n. b. e. of B'/B.*

(2) *The characteristic of $\bar{B}$ is a prime p and the order of G' is a power of p.*

Proof. Let $B = \sum B_0 e_{ij}$, where $\{e_{ij}\}$ is a system of matrix units and $B_0 = V_B(\{e_{ij}\})$ a completely primary ring ([5; Th. 46.9]).[2] If $A_0 = V_A(\{e_{ij}\})$ then A_0/B_0 is G-Galois by [3; Th. 5.8]. Hence, by [1; Th. 2],

1) An element a of A is called a ***root element*** if aA does not contain a non-zero idempotent. Following G. Azumaya, A is called a ***strongly semi-primary ring*** if the set $\mathfrak{N}^*$ of all root elements of A forms an ideal and $A/\mathfrak{N}^*$ is (right or left) Artinian. It is known that a strongly semi-primary ring A is semi-primary and $\mathfrak{N}^*$ coincides with the (Jacobson) radical of A (cf. [5]). Moreover, in [2, p. 100], Y. Miyashita pointed out that a strongly semi-primary ring is nothing but a semi-perfect ring in the sense of H. Bass. A strongly semi-primary ring is called ***strongly primary*** if it is primary. As to other notations and terminologies used in the present note, we follow [6].

2) A completely primary ring was called a local ring in [6].

A_0/B_0 is free G-Galois, which means that A/B is free G-Galois. Accordingly, B'/B is $\bar{G}$-Galois (cf. [3]), and then free by the above argument. Moreover, one may remark that A and B' are semi-primary by [3 ; Prop. 7.3 (2)]. Hence, by [6 ; Th. 2.1], there exists a right G-n. b. e. u, and by a routine computation we can see that $T_{G'}(u)$ is a right $\bar{G}$-n. b. e. of B'/B. We consider here the mapping ψ of GB_R (isomorphic to the group ring GB) onto $\bar{G}B_R$ defined by $(\sum \sigma x_{\sigma R})\psi = \sum \bar{\sigma} x_{\sigma R}$. Then, ψ is a ring homomorphism, and one will easily see that $T_{G'}(u\alpha) = (T_{G'}(u))(\alpha\psi)$ for every $\alpha \in GB_R$. As GB_R is semi-primary by [3 ; Prop. 7.3 (2)] or [5 ; Th. 46.2], $u\alpha$ is again a right G-n. b. e. when and only when α is a unit of GB_R. Similarly, $T_{G'}(u\alpha)$ is a right $\bar{G}$-n. b. e. when and only when $\alpha\psi$ is a unit of $\bar{G}B_R$. Our equivalence is therefore evident by the above proposition.

Corollary. *Let A/B be G-Galois. If B is a strongly primary proper subring of A then the following conditions are equivalent:*

(1) *a is a right G-n. b. e. of A/B whenever $T_G(a)$ is a unit of B.*

(2) *The characteristic of $B/\mathfrak{R}(B)$ is a prime p and the order of G is a power of p.*

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