ON GROUP RINGS OVER SEMI-PRIMARY RINGS

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Throughout the present paper, A will represent a semi-primary ring (with 1), namely, a ring such that the residue class ring $\bar{A}$ of A modulo the Jacobson radical $\mathfrak{R}=\mathfrak{R}(A)$ is (right or left) Artinian, G a multiplicative group of finite order, and H a normal subgroup of G. Moreover, we use the following conventions: $G=\bigcup_{i=1}^{t} H\sigma_i (\sigma_1=1)$ is the right coset decomposition of G modulo H, and $G^*=G/H$. We consider the following commutative diagram of group rings and ring epimorphisms:

$$\begin{array}{ccc} AG & \xrightarrow{\psi_1^*} & AG^* \\ \downarrow{\scriptstyle \bar{\psi}_1} & & \downarrow{\scriptstyle \bar{\psi}_2} \\ \bar{A}G & \xrightarrow{\psi_2^*} & \bar{A}G^* \end{array}$$

where $\bar{\psi}_1: \sum_{\sigma\in G} a_\sigma\sigma \longmapsto \sum_{\sigma\in G}\bar{a}_\sigma\sigma$, $\psi_1^*: \sum_{\sigma\in G} a_\sigma\sigma \longmapsto \sum_{\rho\in G} a_\sigma\sigma^*$ and $\bar{\psi}_2$ and ψ_2^* are similarly defined.[1] We set $\psi=\bar{\psi}_2\psi_1^*=\psi_2^*\bar{\psi}_1$, and further for $H=G$ we write $\phi_0=\psi_1^*$ and $\phi=\psi$, namely, $\phi_0: \sum_{\sigma\in G} a_\sigma\sigma \longmapsto \sum_{\sigma\in G} a_\sigma$ and $\phi: \sum_{\sigma\in G} a_\sigma\sigma \longmapsto \sum_{\sigma\in G}\bar{a}_\sigma$.

Evidently, Ker $\phi_0=\{\sum_{\sigma\in G} a_\sigma\sigma\,;\sum_{\sigma\in G} a_\sigma=0\}$, Ker $\phi=\{\sum_{\sigma\in G} a_\sigma\sigma\,;\sum_{\sigma\in G} a_\sigma\in\mathfrak{R}\}$, Ker $\bar{\psi}_1=\mathfrak{R}\cdot G$, Ker $\psi_1^*=\{\sum_i\sum_{\eta\in H} a_{\eta i}\eta\sigma_i : \sum_{\eta\in H} a_{\eta i}=0$ for $i=1, 2, \cdots, t\}$ and Ker $\psi=\{\sum_i\sum_{\eta\in H} a_{\eta i}\eta\sigma_i\,;\sum_{\eta\in H} a_{\eta i}\in\mathfrak{R}$ for $i=1, 2, \cdots, t\}$.

If ϕ_0' and ϕ' denote respectively the restrictions of ϕ_0 and ϕ to AH, then we obtain Ker $\psi_1^*=\sum_{i=1}^{t}$ Ker $\phi_0'\cdot\sigma_i$ and Ker $\psi=\sum_{i=1}^{t}$ Ker $\phi'\cdot\sigma_i$.

The main theme in the present paper will concern the radical of AG, and our results (Ths. 1—3) will contain [4; Prop. 1], [1; Cor. 1.1] and [2; Th.], respectively.

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1. The following is evident by [3; Th. 46.2], and will be used freely in our subsequent study.

Lemma 1. *AG is semi-primary and the radical $\mathfrak{R}(AG)$ contains $\mathfrak{R}(AH)\cdot G$.*

1) $\bar{a}_\sigma$ means the residue class of $a_\sigma(\in A)$ modulo $\mathfrak{R}$, and σ^* the residue class of σ modulo H.

We shall prove first the following whose proof is quite similar to that of [4; Prop. 1].

Theorem 1. *If* $t=(G:H)$ *is a unit of* A *then* $\mathfrak{N}(AG)=\mathfrak{N}(AH)\cdot G$.

Proof. Let $f: M \longrightarrow N$ be an AG-epimorphism where M and N are (unital) left AG-modules. If $g: N \longrightarrow M$ is an AH-homorphism of N into M such that $fg=1$, then $g^*=(G:H)^{-1}\cdot\sum_{i=1}^{t}\sigma_i^{-1}g\sigma_i$ is an AG-homomorphism of N into M such that $fg^*=1$. Thus, we see that any AG-module is (AG, AH)-projective, and therefore any $AG/\mathfrak{N}(AH)\cdot G$-module is $(AG/\mathfrak{N}(AH)\cdot G, AH/\mathfrak{N}(AH))$-projective. As $AH/\mathfrak{N}(AH)$ is semi-simple, it follows then $AG/\mathfrak{N}(AH)\cdot G$ is semi-simple. Hence, $\mathfrak{N}(AH)\cdot G\supseteq\mathfrak{N}(AG)$.

The next is rather familiar, and will be used in the proof of Th. 2.

Lemma 2. *Let* A *be semi-simple. If* G *is a* p*-group and* $pA=0$ *then* $\mathfrak{N}(AG)=\text{Ker }\phi_0$.

Proof. Evidently, $\text{Ker }\phi_0\supseteq\mathfrak{N}(AG)$. It remains therefore to prove $\mathfrak{N}(AG)\supseteq\text{Ker }\phi_0$. Let the order of G be p^e. We shall prove the assertion by the induction on e. In case $e=1$, it is easy to see that $(\text{Ker }\phi_0)^p=(\sum_{\sigma\in G}A(1-\sigma))^p=0$. Next, suppose $e>1$ and that our assertion is true for $e-1$. As G is a p-group, we can find a normal subgroup H of G such that $(G:H)=p$. By the induction hypothesis, $\text{Ker }\psi_1^*=\mathfrak{N}(AH)\cdot G\subseteq\mathfrak{N}(AG)$. Since AG is Artinian, $\text{Ker }\psi_1^*$ is then nilpotent. Evidently, $\psi_1^*(\text{Ker }\phi_0)\subseteq\mathfrak{N}(AG^*)$, and so $\psi_1^*(\text{Ker }\phi_0)^p=0$ by the case $e=1$, which means that $\text{Ker }\phi_0$ is nlpotent.

Corollary 1. *If* G *is a* p*-group and* $p\bar{A}=0$ *then* $\mathfrak{N}(AG)=\text{Ker }\phi$.

Proof. Evidently, $\text{Ker }\phi\supseteq\mathfrak{N}(AG)$. On the other hand, by Lemma 2, $\bar{\psi}_1(\text{Ker }\phi)\subseteq\mathfrak{N}(\bar{A}G)$, and so we can find some positive integer k such that $(\text{Ker }\phi)^k\subseteq\text{Ker }\bar{\psi}_1=\mathfrak{N}\cdot G\subseteq\mathfrak{N}(AG)$. Hence, we have $\text{Ker }\phi\subseteq\mathfrak{N}(AG)$.

Now, we can prove the following that contains [1; Cor. 1.1].

Theorem 2. *If* G *has a normal* p*-Sylow subgroup* P *and* $p\bar{A}=0$ *then* $\mathfrak{N}(AG)=\mathfrak{N}(AP)\cdot G$.

Proof. By Th. 1, we have $\bar{\psi}_1(\mathfrak{N}(AG))\subseteq\mathfrak{N}(\bar{A}G)=\mathfrak{N}(\bar{A}P)\cdot G$. Hence, by Cor. 1, it follows $\mathfrak{N}(AG)\subseteq\bar{\psi}_1^{-1}(\mathfrak{N}(\bar{A}P)\cdot G)=\mathfrak{N}(AP)\cdot G$. completing the proof.

2. In this section, we shall restrict our attention to the case that A

is a primary ring. The proof of the following will proceed in the same way as in that of [2; Lemma], and may be omitted.

Lemma 3. *Let A be primary, and the order $|G|$ of G greater than 1. If α is a unit of AG whenever $\phi(\alpha)$ is a unit of $\bar{A}$, then $\bar{A}$ is of characteristic $p(\neq 0)$ and G is a p-group.*

Now, we shall give a slight generalization of [2; Th.].

Theorem 3. *If A is primary and $|H|>1$, then the following conditions are equivalent:*

(1) *$\bar{A}$ is of characteristic p, and H is a p-group.*

(2) *AH is primary.*

(3) *α is a unit of AG whenever $\psi(\alpha)$ is a unit of $\bar{A}G^*$.*

(4) *α is a unit of AG whenever $\psi_1^*(\alpha)$ is a unit of AG^*.*

Proof. The implications (3)$\Leftrightarrow$(4) and (1)$\Rightarrow$(2) are easy consequences of Ker $\bar{\psi}_2=\mathfrak{N}\cdot G^*\subseteq\mathfrak{N}(AG^*)$ and $\mathfrak{N}(AH)=$Ker ϕ'(Cor. 1), respectively.

(2)$\Rightarrow$(3): As the radical $\mathfrak{N}(AH)$ of the primary ring AH coincides with Ker ϕ', we have Ker $\psi=\sum$ Ker $\phi'.\sigma_i=\mathfrak{N}(AH)\cdot G\subseteq\mathfrak{N}(AG)$. Hence, there holds (3).

(3)$\Rightarrow$(1): If γ is in Ker ψ then $1-\gamma$ is a unit of AG. Hence, Ker $\phi'\subseteq$ Ker $\psi\subseteq\mathfrak{N}(AG)$. Noting here that $\{\sigma_i\}$ is a free AH-basis of AG, we readily see that Ker ϕ' is a quasi-regular ideal of AH. Hence, by Lemma 3, we obtain (1).

The proof of the next corollary proceeds in the same way as that of the above theorem did.

Corollary 2. *If A is completely primary and $|H|>1$, then the following conditions are equivalent:*

(1) *$\bar{A}$ is of characteristic p, and H is a p-group.*

(2) *AH is completely primary.*

(3) *α is a unit of AG whenever $\psi(\alpha)$ is a unit of $\bar{A}G^*$.*

(4) *α is a unit of AG whenever $\psi_1^*(\alpha)$ is a unit of AG^*.*

Finally, we shall prove the following:

Corollary 3. *If A is a strongly primary ring*[2] *and $|G|>1$ then the following conditions are equivalent:*

(1) *A is a completely primary ring such that $\bar{A}$ is of characteristic*

2) Namely, A is a ring such that $\bar{A}$ is simple and idempotents of $\bar{A}$ can be lifted.

p, and G is a p-group.

(2) *AG is completely primary.*

(3) *AG contains no non-trivial idempotents.*

(4) *α is a unit of AG whenever $\phi(\alpha)$ is non-zero.*

Proof. (1)$\Rightarrow$(3) and (1)$\Rightarrow$(2) are clear by Cor. 2.

(3)$\Rightarrow$(1): Suppose q is a prime factor of $|G|$ different from the characteristic of $\bar{A}$. Now, take a subgroup Q of G whose order is q, and consider the element $\varepsilon = q^{-1}\cdot\sum_{\sigma\in Q}\sigma$. It is easy to see that ε is a non-trivial idempotent.

(2)$\Rightarrow$(4): If $\phi(\alpha)\neq 0$ then α is not contained in Ker $\phi=\mathfrak{N}(AG)$, which means that α is a unit of AG.

(4)$\Rightarrow$(1): By Lemma 3, $\bar{A}$ is of characteristic p and G is a p-group. Hence, $\mathfrak{N}(AG)=$Ker ϕ by Cor. 1. Accordingly, if α is not contained in $\mathfrak{N}(AG)$ then $\phi(\alpha)\neq 0$, and then α is a unit. This means that AG is a completely primary ring.

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