A NOTE ON SCHAUDER BASIS

OSAMU KATSUMATA

Preliminary and notations. Let E denote a topological linear space over the real or complex scalar field. A sequence $(b_n)_{n=1,2,\cdots}$ in E is called a basis of E if for any element x of E there exists a unique sequence $(t_n)_{n=1,2,\cdots}$ of scalars such that $x=\sum_{n=1}^{\infty}t_nb_n$. The convergence of the series is in respect to the vector topology of E. If (b_n) is a basis of E and $x=\sum_{n=1}^{\infty}t_nb_n$, then the uniqueness of the expansion ensures that the correspondence $x \longrightarrow t_n$ is a linear functional for each n. When all of these linear functionals are continuous, the basis $(b_n)_{n=1,2,\cdots}$ is named Schauder basis. E with a basis being complete and metrizable is a sufficient condition for the basis to be Schauder (Arsove [1], p. 368). Also in $\mathscr{LF}$-space (strict inductive limit of Fréchet spaces) a basis is a Schauder basis (Arsove and Edwards [2], p. 113). In this note, we shall show that the some theorems known in Banach spaces also hold in certain Fréchet spaces.

For a toplogical linear space E, let us denote by E' its topological dual. The topologies $\sigma(E, E')$, $\mathcal{T}_b(E, E')$ (abbreviated $\mathcal{T}_b(E)$) are weak-, strong-topologies on E', that is, the topologies deduced from the polars of all finite sets, of all bounded sets of E, respectively. The facts used in this note about topological linear spaces will be found for example in Bourbaki [4], Kelley [7], Köthe [8] or Robertsons [10]. We mean the biorthogonal pair $(x_n, f_n)_{n=1,2,\cdots}$ in (E, E') as the sequences (x_n) in E and (f_n) in E' which satisfy $<x_m, f_n>=\delta_{mn}$ (Kronecker's delta). Let us put

$$(1) \qquad U_m(x)=\sum_{n=1}^{m}<x, f_n>b_n, \qquad V_m(x)=x-U_m(x)$$

for a biorthogonal pair (x_n, f_n) in (E, E'). Then, U_m and V_m are continuous operators in E. Moreover, U_m^* will mean the transpose of U_m defined by the relation $<x, U_m^*(f)>=<U_m(x), f>$ for all $x\in E$, $f\in E'$, and similarly for V_m^*. A sequence (x_n) in E is basic if and only if it is a Schauder basis for the clcsed linear hull (denoted by $[x_n]$) of the set $\{x_n; n=1, 2, \cdots\}$.

Characterization of a basic sequence and its corollaries. In what follows, "completeness" will play the essential role. First, we shall prove a lemma.

Lemma. *Let E be a complete topological liner space. If $(T_n)_{n=1,2,\cdots}$ is an equicontinuous sequence of linear operators which map E into E, then $E_0=\{x;\ \lim_{n\to\infty} T_n(x)$ exists$\}$ is a closed linear subspace of E.*

Proof. It is clear that E_0 is a linear subspace of E from the linearity of each T_n. Let x_0 be any point of the closure of E_0, then a net $(x_\alpha)_{\alpha\in A}$ in E_0 converges to x_0, where for every $\alpha\in A$ there exists $y_\alpha\in E$ such that $\lim_{n\to\infty} T_n x_\alpha=y_\alpha$. For every V, a closed neighbourhood of 0, there exists a neighbourhood U of 0 which satisfies $U+U+\bar{U}\subset V$, and $T_n(U)\subset V$ for each n by equicontinuity. The net $(x_\alpha)_{\alpha\in A}$ converges to x_0, so there exsits $\delta\in A$ such that if $\alpha, \beta\geqq\delta$ then $x_\alpha-x_\beta\in U$ and so $T_n(x_\alpha-x_\beta)\in T_n(U)\subset V$. This and the closedness of V imply $y_\alpha-y_\beta\in V$, $\alpha, \beta\geqq\delta$, i. e. $(y_\alpha)_{\alpha\in A}$ to be a Cauchy net in E. Completeness assumption assures then the existence of an element $y_0\in E$ to which (y_α) converges. The two nets on the same directed system A converge, therefore we can find α_1 in A such that both $y_{\alpha_1}-y_0$ and $x_0-x_{\alpha_1}$ are in U. For this α_1 there exists an integer n_1 such that $T_n(x_{\alpha_1})-y_{\alpha_1}\in\bar{U}$ for all n larger than n_1. Then, $T_n(x_0)-y_0=T_n(x_0)-T_n(x_{\alpha_1})+T_n(x_{\alpha_1})-y_{\alpha_1}+y_{\alpha_1}-y_0\in U+U+\bar{U}\subset V$, $\geqq n_1$. Hence, $T_n(x_0)$ converges to y_0 and we finish the proof.

Theorem. *Let E be a Fréchet space, $(b_n, f_n)_{n=1,2,\cdots}$ a biorthogonal pair in (E, E'), and define $U_m(x)=\sum_{n=1}^{m}<x, f_n>b_n$ for every $x\in E$ and every natural number m. Then, (b_n) is basic if and only if the sequence $(U_m)_{m=1,2,\cdots}$ is pointwise bounded on $E_1=[b_n]$* (A. Wilansky [13], p. 210).

Proof. When (b_n) is basic, the sequence $U_m(x)$ converges to x for every $x\in E_1$ and $\{U_m(x);\ m=1, 2, \cdots\}$ is a bounded set in E. Conversely, E_1 being a closed linear subspace of a Fréchet spece, E_1 is itself a barrelled space. When we regard U_m as a pointwise bounded sequence of continuous linear operators from E_1 to E_1, $\{U_m;\ m=1, 2, \cdots\}$ is equicontinuous. Here we can apply the preceding lemma to obtain a closed linear subspace $\{x\in E_1;\ \lim_{m\to\infty} U_m(x)$ exists$\}$. As it contains all b_n's, it coincides with E_1, that is, $x=\sum_{n=1}^{\infty}<x, f_n>b_n$ for all $x\in E_1$. This implies that (b_n) is a Schauder basis in E_1.

In the following, metrizability of $(E', \mathcal{T}_b(E))$ is needed. This is equivalent to the condition that all bounded sets in E has countable cobase, i. e. there is a countable family of bounded sets such that every bounded set is contained in some member of the family.

Corollary. *Let E be a Fréchet space which has a countable co-base for all bounded sets. If E has a basis $(b_n)_{n=1,2,\cdots}$, then there exists a sequence $(f_n)_{n=1,2,\cdots}$ in E' biorthogonal to (b_n), which is basic in $(E', \mathcal{T}_b(E))$.*

Proof. Because every basis in a Fréchet space is a Schauder basis, there exists a biorthogonal sequence (f_n) in E' to (b_n). If we choose U_m as in (1) then the sequence is equicontinuous by the same reason as in the last part of the proof of the theorem, and the sequence of their transposes U_m^* is equicontinuous in respect to $\mathcal{T}_b(E)$. Next, for every $g \in E'$, $x \in E$ we have

$$<x, U_m^*(g)> = <U_m(x), g> \longrightarrow <x, g>, \text{ as } m \longrightarrow \infty,$$

which means $\{U_m^*(g); m=1, 2, \cdots\}$ is a $\sigma(E, E')$ bounded set. From this and the fact that E is a barrelled space, $\{U_m^*(g); m=1, 2, \cdots\}$ becomes a $\mathcal{T}_b(E)$ bounded set for each $g \in E'$. Now, $(E', \mathcal{T}_b(E))$ is complete as the strong dual of a bornologic space E. We conclude therefore by the preceding theorem, that (f_n) is basic in $(E', \mathcal{T}_b(E))$.

As corollaries to the theorem, we shall mention two propositions.

Proposition 1 (M. M. Day [5], p. 70). *Let E be a Fréchet space which has a countable co-base for all bounded sets, and let $(b_n)_{n=1,2,\cdots}$ be a basis and $(f_n)_{n=1,2,\cdots}$ be a sequence in E' biorthogonal to (b_n). The following conditions are equivalent:*

(i) *(f_n) is a $\mathcal{T}_b(E)$ Schauder basis.*

(ii) *$V_m^* \longrightarrow 0$, as $m \longrightarrow \infty$, in respect to the strong operator topology.*

(iii) *For every $f \in E'$ and for every bounded set B of E,* $\sup \{|f(x)|; x \in B \cap [b_n, b_{n+1} \cdots]\} \longrightarrow 0$, *as $n \longrightarrow \infty$.*

(iv) *$[f_n] = E'$ i.e. $\{f_n; n=1, 2, \cdots\}$ is fundamental in $(E', \mathcal{T}_b(E))$.*

Remark. (a) V_m^* means the transpose of V_m where V_m is defined as in (1). When for every $f \in E'$ $V_m^*(f) \longrightarrow 0$ in respect to $\mathcal{T}_b(E)$, we say that $V_m^* \longrightarrow 0$ in respect to the strong operator topology.

(b) A Schauder basis in a Banach space is called shrinking when it satisfies the condition that for every $f \in E'$,

$$\sup \{|f(x)|; x \in [b_n, b_{n+1}, \cdots], \|x\| \leqq 1\} \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

The property "shrinking" is related to the basis of a reflexive Banach space (cf. for instance James [6], Retherford [9] and Singer [12]).

(c) For a barrelled space X (in particular, for a Fréchet space), it can be proved that if X' provides the toplogy of uniform convergence of all precompact sets then every sequence biorthogonal to a Schauder basis spans X'. Comparing this with the condition (iv), one will see the difference between a shrinking basis and a Schaüder basis.

Proof. (i)⇒(ii): For every $x\in E, f\in E'$,

$$<x, V_m^*(f)> = <V_m(x), f> = <x-\sum_{n=1}^m <x, f_n> b_n, f>$$
$$= <x, f-\sum_{n=1}^m <b_n, f> f_n>.$$

Accordingly, if (f_n) is a $\mathcal{T}_b(E)$-basis then for every bounded set B of E we have $\sup_{x\in B} |<x, V_m^*(f)>| \longrightarrow 0\ (m\longrightarrow\infty)$, that is, $V_m^* \longrightarrow 0\ (m\longrightarrow\infty)$ in respect to the strong operator topology.

(ii)⇒(iii): Since (b_n) is a Schauder basis, it is obvious that x is in $B\cap[b_n, b_{n+1}, \cdots]$ when and only when $x=V_n(x)$ and $x\in B$. On the other hand, for every $f\in E'$

$$\sup\{|<x, f>|; x=V_n(x), x\in B\} \leqq \sup\{|<V_n(x), f>|; x\in B\}$$
$$=\sup\{|<x, V_n^*(f)>|: x\in B\}.$$

We see therefore (ii) implies (iii).

(iii)⇒(iv): For every $f\in E'$ and every bounded set B of E, if we set $B_1=\bigcup_{m=1}^\infty V_m(B)$ then there holds

$$\sup_{x\in B}|<x, \sum_{n=1}^m <b_n, f> f_n - f>| = \sup_{x\in B}|<V_m(x), f>|$$
$$\leqq \sup_{x\in B_1}\{|<x, f>|; x=V_m(x)\}.$$

(Note that $V_m^2=V_m$.) As the union B_1 of the images of bounded set B by equicontinuous operators is a bounded set of E, this yields the implication (iii)⇒(iv).

(iv)⇒(i): This is a consequence of the preceding corollary.

Proposition 2. (I. Singer [11], p. 77). *Let E be a Fréchet space which has a countable co-base for all bounded sets. If E' contains a $\sigma(E, E')$ Schauder basis $(f_n)_{n=1,2,\cdots}$ then E contains a Schauder basis $(b_n)_{n=1,2,\cdots}$, and conversely.*

Proof. For the $\sigma(E, E')$-Schauder basis (f_n) there exists a biorthogonal sequence of $\sigma(E, E')$-continuous linear functionals (b_n), which we may regard as a subset of E. Since $U_m^*(f)$ $\sigma(E, E')$-converges to f, it follows that $U_m(x)$ $\sigma(E', E)$-converges to x. Accordingly, $\{U_m(x); m=1, 2, \cdots\}$ is $\sigma(E', E)$-bounded, i.e. it is a bounded set of E. Now, noting that the linear hull of b_n's is dense in E, the same argument used in the proof of the theorem shows that the sequence (b_n) is a Schauder basis. Conversely, assume that (b_n) is a Schauder basis in E. Then, by definition, there exists a biorthogonal sequence (f_n) and for each $x \in E$ there holds $x = \sum_{n=1}^{\infty} <x, f_n> b_n$. As $<x, f - \sum_{n=1}^{m} <b_n, f> f_n> = <x - \sum_{n=1}^{m} <x, f_n> b_n, f>$ for every $f \in E'$, we readily see that $\sum_{n=1}^{m} <b_n, f> f_n$ $\sigma(E, E')$-converges to f. Moreover, b_n's being continuous linear functionals on $(E', \sigma(E, E'))$ and biorthogonal to (f_n), the expansion is unique.

REFERENCES

[1] M. G. ARSOVE: The Paley-Wiener theorem in metric linear spaces, Pacific Journal of Mathematics, 10 (1960), 365—379.

[2] M. G. ARSOVE and R. E. EDWARDS: Generalized bases in topological linear spaces, Studia Mathematica, 19 (1960), 95—113.

[3] S. BANACH: Théorie des opérations linéaires, Warszawa, 1933.

[4] N. BOURBAKI: Elements de Mathématique, Espaces vectoriels topologiques, Hermann, Paris, 1955.

[5] M. M. DAY: Normed linear spaces, Springer-Verlag, 1962.

[6] R. C. JAMES: Bases and reflexivity of Banach spaces, Annals of Mathematics, 52 (1950), 518—527.

[7] J. L. KELLEY and COAUTHORS: Linear topological spaces, D. Van Nostrand, New York, 1963.

[8] G. KÖTHE: Topologische lineare Räume I, Springer-Verlag, 1960.

[9] J. R. RETHERFORD: Bases, basic sequences and reflexivity of linear topological spaces, Mathematische Annalen, 164 (1966), 280—285.

[10] A. ROBERTSON and W. ROBERTSON: Topological vector spaces, Cambridge University Press, 1964.

[11] I. SINGER: Weak* bases in conjugate Banach spaces, Studia Mathematica, 21 (1961), 75—81.

[12] I. SINGER: Basic sequences and reflexivity of Banach spaces, Studia Mathematica, 21 (1962), 351—369.

[13] A. WILANSKY: Functional analysis, Blaisdell Publishing Company, 1964.

DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

(*Received April 17, 1969*)