A NOTE ON SEPARABLE EXTENSIONS OF COMMUTATIVE RINGS

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Commutative separable algebras have been studied in [1] and [2], [3] where the main ideas are based on the theory of fields. Moreover, in [4], G. J. Janusz presented a number of explicit results for commutative separable algebras. In this paper, we shall make a remark on commutative separable algebras (Theorem), where this paper depends heavily on [4].

Throughout the present paper, A will be a commutative ring with the identity element 1, and B a subring of A containing the identity element 1 of A. As in [4], if A is projective, and separable over B then A will be called a strongly separable B-algebra. By [5, Villamayor's Theorem], a strongly separable B-algebra is a finitely generated B-module. Moreover, by [4, Th. 1.1], a strongly separable B-algebra with no proper idempotents (no idempotents except 0 and 1) can be imbeded in a Galois extension of B with no proper idempotents. If N is a Galois extension of B then the Galois group will be denoted by $G(N/B)$. Our purpose of this paper is to prove the following theorem.

Theorem. *Let $A \supsetneqq B$, and A a strongly separable B-algebra without proper idempotents. Then, $B[a]$ is separable over B for every $a \in A$ if and only if A is a field.*

Firstly, we shall prove the following

Lemma. *Let $B[a]$ be a commutative ring without proper idempotents, and $a \notin B$. If $B[a]$ is a strongly separable B-algebra then a is not nilpotent.*

Proof. Let N be a ring extension of $B[a]$ without proper idempotents which is Galois over B. Then, there exists an element σ of $G(N/B)$ such that $a \neq a^{\sigma} = b$. By [4, Lemma 2.7 and Lemma 2.1], $a-b$ is an inversible element of N. If $a^n = 0$ for some natural number n then $b^n = 0$, and so, we have a contradiction

$$0 \neq (a-b)^{2n} = \sum_{i=0}^{2n}(-1)^i \binom{2n}{i} a^{2n-i} b^i = 0$$

Hence a is not nilpotent.

Now, we shall prove our theorem.

The proof of Theorem. Let N be a ring extension of A without proper idempotents which is Galois over B. If A is a field then, for every nonzero element a of B, we have $(a^{-1})^{\sigma} = (a^{\sigma})^{-1} = a^{-1}$ for all $\sigma \in G(N/B)$, which implies $a^{-1} \in B$; hence B is a field, and A is separable over B in the sense of classical separability, and so, every element of A is separable over B. Conversely, we assume that $B[a]$ is separable over B for every $a \in A$. At first, we shall prove that B is a field. Let a be an element of A which is not contained in B. Then, there exists an element σ of $G(N/B)$ such that $a \neq a^{\sigma}$. By [4, Lemma 2.7 and Lemma 2.1], $a - a^{\sigma}$ is an inversible element of N. Let b be a nonzero element of B. Then $b(a-a^{\sigma}) \neq 0$. Hence $ba \neq (ba)^{\sigma}$, and so, $ba-(ba)^{\sigma} = b(a-a^{\sigma})$ is an inversible element of N. Therefore b is an inversible element of N. Since $(b^{-1})^{\tau} = (b^{\tau})^{-1} = b^{-1}$ for all $\tau \in G(N/B)$, we have $b^{-1} \in B$. Thus B is a field. Since A is a finitely generated B-module, A is an Artinian ring. By lemma, the radical of A is zero. Hence A is a semisimple ring. Noting that A has no proper idempotents, A is a field.

The following corollaries are direct consequences of our theorem.

Corollary 1. *Let $A \supsetneqq B$, and A a strongly separable B-algebra without proper idempotents. If A is not a field then there exists an intermediate ring $B[a]$ of A/B such that $B[a]$ is not separable over B.*

Corollary 2. *Let A be a strongly separable B-algebra without proper idempotents. If $B[a]$ is separable over B for every $a \in A$ then, for every intermediate ring D of A/B, $D = B[d]$ for some element d of A.*

Corollary 3. *Let $A \supsetneqq B$ and suppose A has no proqer idempotents. Then, $B[a]$ is strongly separable over B for every $a \in A$ if and only if A is a field which is algebraic and separable over B.*

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