A NOTE ON NETS

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For an accumulation point of a topological space we define a class of directed sets which can be domains of non-constant nets having this point as a cluster point. For certain accumulation points we also study the existence of a convergent net with prescribed domain.

We adopt the terminology and notation used in [2]. Throughout, unless otherwise specified, P will denote a general topological space.

If x is an accumulation point of P, then there is a net $\{x_\alpha\}_{\alpha\in D}$ in $P-(x)$ which converges to x (see [2], p. 66). However, the directed set D on which the net $\{x_\alpha\}_{\alpha\in D}$ is defined cannot be completely arbitrary. In this note we give a simple condition on D which is sufficient for the existence of a net $\{x_\alpha\}_{\alpha\in D}$ in $P-(x)$ having x as a cluster point (see [2], p. 71). Loosely speaking, this condition requires that the set D must be "longer" than a local base at x (see [2], p. 50). In the special case when the point x has a nested local base (see [1], p. 90, def. 4. 12. 4 or later in this paper), we give also a sufficient condition for the existence of a net $\{x_\alpha\}_{\alpha\in D}$ in $P-(x)$ converging to x.

We begin with three simple examples which will illustrate the problem.

1. Example. Let P be the set of all ordinals less than or equal to the first uncountable ordinal Ω and let D be a countable directed set. In the order topology Ω is an accumulation point of P. However, every net defined on D which is in $P-(\Omega)$ is already in $P-U$, where U is a neighborhood of Ω. Apparently, in this case, the set D was "too short."

2. Example. Let $P=[0, 1]$ and let D be the well ordered set of all countable ordinals. Suppose $\{x_\alpha\}_{\alpha\in D}$ is a net in $P-(0)$ converging to 0. Then there are ordinals $\alpha_n\in D$, $n=1, 2, \cdots$, such that $x_\alpha<1/n$ for all ordinals $\alpha\in D$ for which $\alpha>\alpha_n$. Choose an ordinal $\alpha_0\in D$ larger than all α_n. It follows that $x_{\alpha_0}=0$; a contradiction. Here the set D was "to long". However, we can still define a net $\{x_\alpha\}_{\alpha\in D}$ in $P-(0)$ which has 0 as its cluster point. Indeed, it suffices to let $x_\alpha=1/(n+1)$, where $\alpha\in D$ and n is the number of isolated ordinals between α and the largest limit ordinal less than or equal to α.

3. Example. Let P and D be the same as in example 2 and let N be the well ordered set of all finite ordinals. The product directed set $D\times N$ (see [2], p. 68) is "quite long," e. g., it has no countable cofinal subset. Yet, letting $x_{(\alpha,n)}=1/(n+1)$ for all $(\alpha, n)\in D\times N$, we have defined a net in $P-(0)$ which converges to 0.

If $(D, >)$ is a directed set and if $\alpha\in D$, we put $D_\alpha=\{\beta\in D: \beta>\alpha\}$. Obviously, D_α is a cofinal subset of D. We recall that every local base Γ at $x\in P$ is directed by the inclusion $\subset$.

4. Definition. A directed set D is said to be *sufficiently long* for a point $x\in P$ if and only if for every $\alpha\in D$ there is a local base Γ_α at x which can be isotonically embedded into D_α.

Let x be an accumulation point of P which has a countable local base. Then a directed set D is sufficiently long for x if and only if D has no maximal element.

If x is an accumulation point of P, we denote by ψ_x a choice function which to every neighborhood U of x associates a point $\psi_x(U)$ from $U-(x)$. If a directed set $(D, >)$ is sufficiently long for x, let $\varphi_\alpha:(\Gamma_\alpha, \subset)\rightarrow(D_\alpha, >)$, $\alpha\in D$, be isotonic embeddings whose existence is required by definition 4.

5. Theorem. *Let x be an accumulation point of P and let $(D, >)$ be a directed set which is sufficiently long for x. Then there is a net $\{x_\alpha, \alpha\in D, >\}$ in $P-(x)$ having x as its cluster point.*

Proof. By $\mathfrak{M}$ we denote the family of graphs of all maps $f: D^f\rightarrow P-(x)$, $D^f\subset D$, which are frequently in every neighborhood U of x (this has meaning, for D^f is a partially ordered set). Since D is sufficiently long for x, we can take $\alpha\in D$ and define $D^g=\varphi_\alpha(\Gamma_\alpha)$ and $g=\psi_x\varphi_\alpha^{-1}: D^g\rightarrow P-(x)$. Given a neighborhood U of x and $\beta\in D^g$, there is $V\in\Gamma_\alpha$ such that $V\subset U\cap\varphi_\alpha^{-1}(\beta)$. Letting $\gamma=\varphi_\alpha(V)$ we obtain $\gamma>\beta$ and $g(\gamma)\in U$. Therefore, the graph of g belongs to $\mathfrak{M}$. It can be readily verified that the union of every non-empty nest $\mathfrak{M}_0\subset\mathfrak{M}$ belongs to $\mathfrak{M}$. By [2], p. 33, thm. 25 (a) there is a maximal element ϕ in $\mathfrak{M}$. Let ϕ be the graph of the map $f: D^f\rightarrow P-(x)$.

Suppose there is an $\alpha\in D-D^f$. If $\alpha<\beta$ for some $\beta\in D^f$, we let $D^h=D^f\cup(\alpha)$, $h(\gamma)=f(\gamma)$ for $\gamma\in D^f$, and $h(\alpha)=f(\beta)$. Then the graph of h belongs to $\mathfrak{M}$ which contradicts the maximality of ϕ. Therefore, $D^f\cap D_\alpha=\emptyset$. Since D is sufficiently long for x, we can define $D^h=D^f\cup\varphi_\alpha(\Gamma_\alpha)$,

$h(\beta)=f(\beta)$ for $\beta\in D^f$ and $h(\beta)=\psi_{\varphi_\alpha}[\varphi_\alpha^{-1}(\beta)]$ for $\beta\in\varphi_\alpha(\Gamma_\alpha)$. As in the first part of the proof it can be shown that the graph of h belongs to $\mathfrak{M}$. This again contradicts the maximality of ϕ.

We conclude that $D^f=D$ and the proposition is proved; for it suffices to let $x_\alpha=f(\alpha)$ for all $\alpha\in D$.

6. Corollary. *Let x be an accumulation point of P and let D be a directed set. Suppose that P is a T_1-space and that x has a countable local base. Then a net $\{x_\alpha\}_{\alpha\in D}$ in $P-(x)$ having x as its cluster point exists if and only if D has no maximal element.*

The next example shows that, in general, a converse of theorem 5 is not correct.

7. Example.[1] Let N be the set of all finite ordinals with the discrete topology and let $\beta(N)$ be the Stone-Čech compactification of N (see [2], p. 153). Choose $x\in\beta(N)-N$ and introduce the relative topology into $P=N\cup(x)$. Then P is a Tychonoff space (see [2], p. 117) and x is an accumulation point of P which has no countable local base. Therefore, naturally ordered, the set N is not sufficiently long for x. Yet, the net $\{n\}_{n\in N}$ is in $P-(x)$ and has x as its cluster point.

A local base for a point $x\in P$ is called *nested* if it is completely ordered by inclusion. A point $x\in P$ is called nested whenever it has a nested local base. Obviously, every point which has a countable local base is nested. Some basic properties of nested points are discussed in [1], sec. 4. 12.

8. Definition. A directed set $(D, >)$ is said to be *suitably long* for a nested point $x\in P$ if and only if there is a nested local base Γ at x and an isotonic embedding $\varphi:(\Gamma, \subset)\to(D, >)$ such that $\varphi(\Gamma)$ has no upper bound in D.

Let x be an accumulation point of P which has a countable local base. Then a directed set D is suitably long for x if and only if it contains a countable unbounded chain (see [2], p. 32). In this case D is also sufficiently long for x. However, in general, a directed set which is suitably long for some point need not be sufficiently long for this point. We give an example.

9. Example. Let P be the same as in example 1 and let $\mathscr{D}$ be the

1) This example was suggested by C. R. Borges.

family of all finite subsets of $P-(\Omega)$. The symbol $>$ is used to denote the ordering in P. For $A, B\in\mathcal{D}$ we set $A\succ B$ if and only if $B\subset A$ or $B=(\beta)$ is a singleton and $\beta<\alpha$ for some $\alpha\in A$. It is easy to see that $\succ$ is a direction on $\mathcal{D}$. The family $\Gamma=\{P_\alpha:\alpha\in P-(\Omega)\}$ is a local base at Ω and the map $\varphi:(\Gamma,\subset)\to(\mathcal{D},\succ)$, defined by the rule $\varphi(P_\alpha)=(\alpha)$ for all $\alpha\in P$, is an isotonic embedding. Since $\varphi(\Gamma)$ has no upper bound in $\mathcal{D}$, the directed set $(\mathcal{D},\succ)$ is suitably long for Ω. On the other hand, if $A\in\mathcal{D}$ is not a singleton, then $\mathcal{D}_A$ contains only countable chains. Hence $(\mathcal{D},\succ)$ is not sufficiently long for Ω.

10. Proposition. *Let x be a nested accumulation point of P and let $(D, >)$ be a directed set which is suitably long for x. Then there is a net $\{x_\alpha, \alpha\in D, >\}$ in $P-(x)$ converging to x.*

Proof. Let Γ and φ be as in definition 7. Since $\varphi(\Gamma)$ has no upper bound in D, the sets $\Gamma_\alpha=\{U\in\Gamma:\alpha\notin D_{\varphi(U)}\}$ are non-empty for all $\alpha\in D$. Denote by f a choice function which to every $\alpha\in D$ associate an element $f(\alpha)$ from Γ_α and put $x_\alpha=\psi_{\Gamma x}[f(\alpha)]$ for all $\alpha\in D$. Given a neighborhood U of x, there is a $V\in\Gamma$ such that $V\subset U$. Let $\alpha\in D$ and $\alpha>\varphi(V)$. If $f(\alpha)\supset V$, then $\varphi[f(\alpha)]<\varphi(V)<\alpha$ which contradicts the definition of f. Therefore $f(\alpha)\subset V$ and hence $x_\alpha\in U$.

11. Corollary. *Let x be an accumulation point of P and let $(D, >)$ be a directed set. Suppose that P is a T_1-space and that x has a countable local base. Then a net $\{x_\alpha, \alpha\in D, >\}$ in $P-(x)$ converging to x exists if and only if D contains a countable unbounded chain.*

If f is a real-valued function of one real variable and if t is a real number then the set of all numbers $s=\lim f(t_n)$ where $\lim t_n=t$ is closed. The following generalizes this fact.

Let $f: P\to Q$ be a mapping of P into a topological space Q, let x be an accumulation point of P, and let Γ be a local base at x. We denote by Q^x the set of all cluster points of all nets $\{f(x_\alpha)\}$ where $\{x_\alpha\}$ is an arbitrary net in P converging to x. We denote by Q^Γ the set of all cluster points of all nets $\{f(x_U)\}_{U\in\Gamma}$ where $\{x_U\}_{U\in\Gamma}$ is a net in P defined on Γ and converging to x.

12. Proposition. *The set Q^x is always closed. If Γ is sufficiently long for every $y\in Q$, then also the set Q^Γ is closed.*

Proof. Let y be an accumulation point of Q^x and let Λ be a local base at y. Choose $U\in\Gamma$ and $V\in\Lambda$. Since $V\cap Q^x\neq\emptyset$, there is a net $\{x_\alpha\}$ in P which is eventually in U and such that $\{f(x_\alpha)\}$ is frequently in V. Hence for some α, $x_\alpha\in U$ and $f(x_\alpha)\in V$. We let $x_{(U,V)}=x_\alpha$ and denote by $>$ the product ordering in $\Gamma\times\Lambda$ (see [2], p. 68). Then the nets $\{x_{(U,V)}, (U, V)\in\Gamma\times\Lambda, >\}$ and $\{f(x_{(U,V)}), (U, V)\in\Gamma\times\Lambda, >\}$ converge to x and y, respectively.

Let Γ be sufficiently long for every $y\in Q$ and let y be an accumulation point of Q^Γ. By the same method as in the proof of Theorem 5 we can construct a net $\{V_U, U\in\Gamma, \subset\}$ of neighborhoods of y which is frequently contained in every neighborhood of y. Choose $U\in\Gamma$. Since $Q^\Gamma \cap V_U\neq\emptyset$, there is a net $\{x_W\}_{W\in\Gamma}$ in P which is eventually in U and such that $\{f(x_W)\}_{W\in\Gamma}$ is frequently in V_U. Hence for some $W\in\Gamma$, $x_W\in U$ and $f(x_W)\in V_U$. If we let $x_U=x_W$, then $\{x_U\}_{U\in\Gamma}$ converges to x and $\{f(x_U)\}_{U\in\Gamma}$ has y as its cluster point.

13. Corollary. *If Q is first countable then Q^Γ is closed.*

The next example shows that the assumption of the second part of proposition 12 cannot be omitted.

14. Example. Let N be the set of all finite ordinals, Q the set of all ordinals less than or equal to the first uncountable ordinal Ω, and let $Q'=Q-(\Omega)$. In Q we consider the order topology. In $P=(Q'\times N)\cup(\infty)$ the points from $Q'\times N$ are isolated and $\Gamma=\{(Q'\times N_n)\cup(\infty)\}_{n=0}^{\infty}$ is a local base at ∞. Thus Γ is not sufficiently long for Ω. If we set $f(\alpha, n)=\alpha$ for every $(\alpha, n)\in Q'\times N$ and $f(\infty)=0$, then $Q^\Gamma=Q'$ which is not closed.

REFERENCES

[1] E. Čech: Topologické prostory, ČSAV, Prague, 1959.
[2] J. L. Kelley: General topology, Van Nostrand, New York, 1955.

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