# A NOTE ON THE DIFFERENTIAL EQUATIONS WITH RELAY-HYSTERESIS

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## § 1. Introduction.

In many cases the control mechanisms contain elements which have the nonlinearities with relay-type characteristics, and consequently the characteristic function $\varphi(\sigma)$ of them has the discontinuities at switching points where $\sigma$ is the so-called feedback signal. On several occasions, for the convenience of treatment, this characteristic function $\varphi(\sigma)$ is given by a step function as Fig. 1.

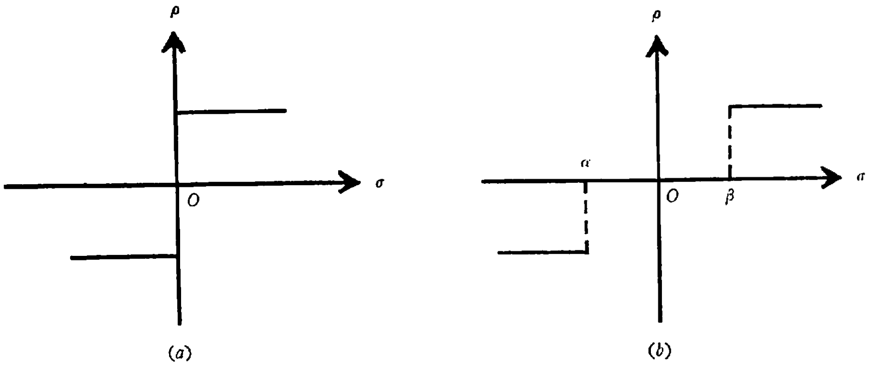


Fig. 1.

In practical questions, the relays do not behave to work on switching accurately as in Fig. 1, and they have somewhat inclination to work in retard. Namely, when $\sigma$ is increasing from some negative value of $\sigma$, the relay operates switching after the lapse of some time-interval, rather than in an instant of passing through the prepared switching point, inversely when $\sigma$ is decreasing from some positive value of $\sigma$, the same phenomenon is observed. Such being the circumstance, we may characterize such inclination as the so-called relay-hysteresis (Fig. 2). Beside his phenomenon, we often encounter the questions of physical systems with relay-hysteresis (see Я. З. Пыпкин [5]).

Strictly speaking, the characteristic function $\varphi(\sigma)$ with relay-hysteresis is not a single-valued function in the mathematical sense, so we must define its meaning. Recently, В. А. Якубович [6] treated this problem

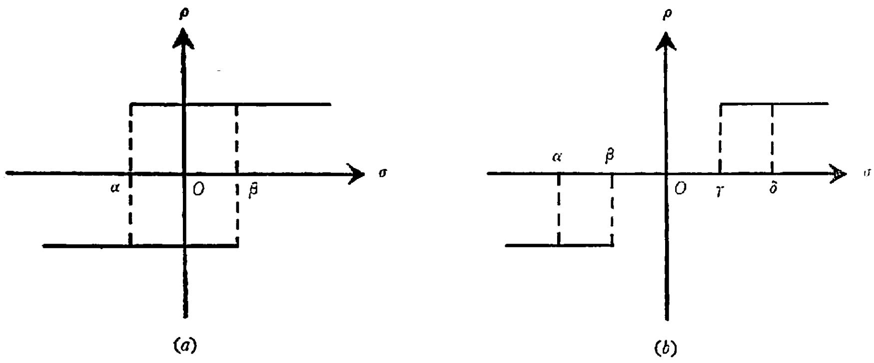


Fig. 2.

and he defined the hysteresis function $\varphi(\sigma)$ as a family of operators which transform a continuous function $\sigma(t)$ to a piece-wise continuous function $\varphi[\sigma; \varphi_0]_t$ where $\varphi_0$ denotes an initial value of hysteresis-curves. Moreover, J. André and P. Seibert [1, 2] discussed the differential equations with discontinuous right hand in connection with the discontinuous control system and they systematically classified the points on the switching lines.

In this note, we are mainly concerned with the behavior of the solutions of differential equations with relay-hysteresis in the neighborhood of the switching lines, so that we prefer to consider that several regions on the phase plane, on each of which differential equations are defined, are connected with the switching lines and the path is continued across the switching line to another region so as the path is continuous in the whole. Especially, we shall try to extend the theory due to J. André and P. Seibert to our problem and apply it to certain differential equations with relay-hysteresis.

## § 2. Construction of the phase plane,

In the following, we consider the system of two differential equations with relay-hysteresis as follows:

$$(2.1)\qquad \begin{cases} \dot{x} = X(x, y, \varphi(\sigma)), \\ \dot{y} = Y(x, y, \varphi(\sigma)), \\ \sigma = \sigma(x, y) \end{cases}$$

where $X(x, y, \rho)$ and $Y(x, y, \rho)$ are continuously differentiable functions with respect to $(x, y, \rho)\in R^3$, $\sigma(x, y)$ is a twice continuously differentiable function with respect to $(x, y)\in R^2$ and $\varphi(\sigma)$ is a hysteresis function. For simplicity, we consider the relay-type hysteresis function $\varphi(\sigma)$ with one loop of the width of hysteresis $\beta-\alpha>0$ as in Fig. 3. Therefore, the graph of $\varphi(\sigma)$ consists of two curves $C_1$ and $C_2$ which are prescribed by continuously differentiable functions $\rho=\varphi_1(\sigma)$ for $\alpha\leqq\sigma<+\infty$ and $\rho=\varphi_2(\sigma)$ for $-\infty<\sigma\leqq\beta$ respectively. We call the points $\sigma=\alpha$ and $\sigma=\beta$ on the $\sigma$-line the switching points. In many cases, the graph of $\varphi(\sigma)$ has discontinuities at the switching points.

Now, we consider the meaning of solutions of such differential equations. We divide the phase plane into three regions, namely,

$$\begin{cases} Q_1=\{(x,y)\,;\,\beta\leqq\sigma(x,y)\}, \\ R=\{(x,y)\,;\,\alpha\leqq\sigma(x,y)\leqq\beta\}, \\ Q_2=\{(x,y)\,;\,\sigma(x,y)\leqq\alpha\}. \end{cases}$$

In the following, we call the curves $H_1:\sigma(x,y)=\alpha$ and $H_2:\sigma(x,y)=\beta$ in the phase plane the switching lines. Moreover, we assume that $\sigma_x$ and $\sigma_y$ do not vanish simultaneously on the switching lines.

Here, we put

$$\begin{cases} S_1=Q_1\cup R, \\ S_2=Q_2\cup R, \end{cases}$$

then the regions $S_1$ and $S_2$ are overlapped on the layer $R$. Therefore, we consider that the differential equations (2.1) have the meaning as follows: On the region $S_1$ there are defined differential equations

$$\text{(2.2)}\qquad \begin{cases} \dot{x}=X(x,y,\varphi_1(\sigma(x,y))), \\ \dot{y}=Y(x,y,\varphi_1(\sigma(x,y))), \end{cases}$$

and on the region $S_2$ there are defined differential equations

$$\text{(2.3)}\qquad \begin{cases} \dot{x}=X(x,y,\varphi_2(\sigma(x,y))), \\ \dot{y}=Y(x,y,\varphi_2(\sigma(x,y))), \end{cases}$$

and we assume that a path on $S_1$ is only continued across the switching line $H_1$ into the region $S_2$, in the other hand, a path on $S_2$ is only continued across the switching line $H_2$ into the region $S_1$.

In the sequel, we may consider that the phase plane consists of two sheets $S_1$ and $S_2$ overlapped on the layer $R$ and the representative point on $S_1$ is only possible to pass through across the boundary $H_1$ to the other sheet $S_2$, inversely, the representative point on $S_2$ is only possible to enter through across the boundary $H_2$ to the sheet $S_1$ (Fig. 4).

In what follows, we shall limit our considerations to the class of the system (2.1) satisfying the following conditions:

1° Any critical point of the differential equations of (2.2) and (2.3) does not lay itself on the switching lines $H_1$ and $H_2$.

2° Each of paths only isolatedly meets with the switching lines $H_1$ and $H_2$, in other words, there no exists any accumulation point of the inter-

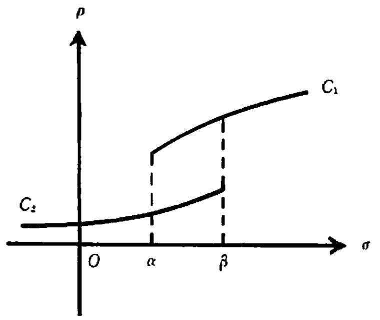

Fig. 3.

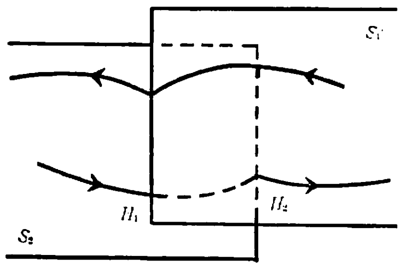

Fig. 4

section of paths with the switching lines.

Such a system we may call admissible.

## § 3. Behavior of paths on the sheet $\tilde{S}_1$,

For the later discussion, it is convenient to consider a region $\tilde{S}_1$ which contains the region $S_1$, say,

$$\tilde{S} = \{(x, y)\,;\, \sigma(x, y) \geqq \alpha - \varepsilon,\ \varepsilon > 0\} \supset S_1,$$

and we assume that the given differential equations of (2.2) are defined in the extended region $\tilde{S}_1$.

In the following, we investigate what happens in the neighborhood of the switching line $H_1$, when the path $\gamma$ on $S_1$ approaches to a point $P(x_0, y_0)$ on the line $H_1$ at a finite time.

On such a question, J. André and P. Seibert discussed in detail and they classified the points on the switching lines. In the following, we make researches on the aspect of paths by the method due to them.

Let

$$\begin{cases} x = x_1(t, x_0, y_0), \\ y = y_1(t, x_0, y_0) \end{cases}$$

be the solution of the equations (2.2), which are defined in the extended region $\tilde{S}_1$, with the initial values $x = x_0, y = y_0$ at the time $t = 0$ where $(x_0, y_0) \in H_1$. Since our system is admissible, the following limits

$$\begin{cases} e_1 = \lim\limits_{t \to -0} \operatorname{sgn}\{\sigma_1(t) - \alpha\}, \\ e_2 = \lim\limits_{t \to +0} \operatorname{sgn}\{\sigma_1(t) - \alpha\} \end{cases}$$

always exist where

$$(3.1) \qquad \sigma_1(t) = \sigma(x_1(t, x_0, y_0), y_1(t, x_0, y_0))$$

and sgn $\sigma$ is the signum function defined by

$$\operatorname{sgn} \sigma = \begin{cases} \dfrac{|\sigma|}{\sigma} & \text{for } \sigma \neq 0, \\ 0 & \text{for } \sigma = 0. \end{cases}$$

In the region $S_1 - H_1$, the values of $\sigma(x, y) - \alpha$ are positive and in the region $\tilde{S}_1 - S_1$ the values of $\sigma(x, y) - \alpha$ are negative, so that we often call the region $S_1 - H_1$ the $(+)$—side of the sheet $\tilde{S}_1$ with respect to the line $H_1$ and we call the region $\tilde{S}_1 - S_1$ the $(-)$—side of $\tilde{S}_1$ with respect to $H_1$.

Under these assumptions, according to the values of $e_1$ and $e_2$, there occur four cases.

Case $(A)$: $e_1 = +1$ and $e_2 = -1$.

In this case, for a sufficiently small time-interval $I$: $-t_0 < t < 0$, the path $\gamma$ is contained in the $(+)$—side of $S_1$ and it tends to the point $P(x_0, y_0)$ at $t$ tends to $0$ increasingly and after the touching with the point $P$, the path passes over the line $H_1$ to the $(-)$—side of the regions $S_1$ as $t$ is increasing (Fig. 5).

Case $(B)$: $e_1 = -1$ and $e_2 = +1$.

In this case, the same as the case (A) occurs but as $t$ changes to $-t$ (Fig. 6).

Case $(C)$: $e_1 = +1$ and $e_2 = +1$.

In this case, the path for $-t_0 < t < 0$ which is contained in the $(+)$—side of $S_1$ tends to the point $P$ as $t$ tends to $0$ increasingly and after the touching the path $\gamma$ remains to stay in the $(+)$ side of $S_1$ without traversing the line $H_1$ (Fig. 7).

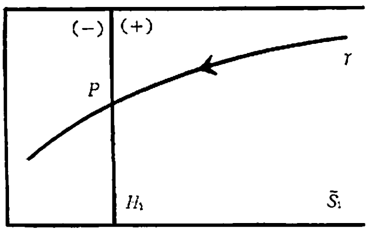


Fig. 5.

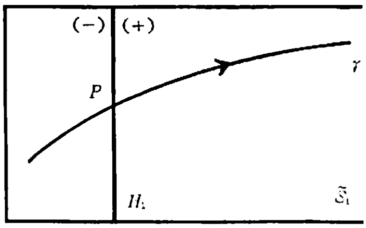


Fig. 6.

Case $(D)$: $e_1 = -1$ and $e_2 = -1$.

In this case, the aspect is the same as the case $(C)$ except that the path $\gamma$ is contained in the $(-)$—side of $S_1$ for $t \neq 0$. Therefore, the solution corresponding to $\gamma$ is not a solution of the initial equations defined on the region $S_1$ with the exception of the point $P$ (Fig. 8).

Now, we consider the conditions under which the above cases occur. We put as follows:

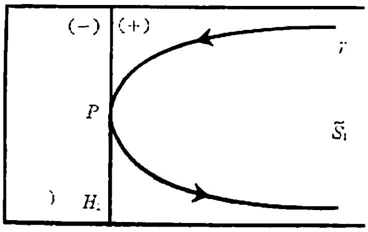


Fig. 7.

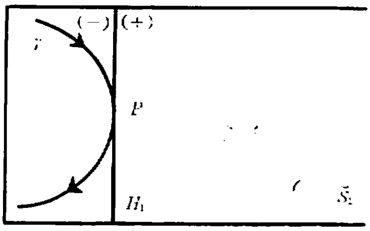


Fig. 8.

$$\begin{cases} F_1(x, y) = X(x, y, \varphi_1(\sigma(x, y))), \\ G_1(x, y) = Y(x, y, \varphi_1(\sigma(x, y))), \end{cases}$$

and we have

$$\begin{cases} \sigma'(0) = \sigma_x F_1 + \sigma_y G_1 \Big|_{\substack{x=x_0 \\ y=y_0}} \\ \sigma_1''(0) = \sigma_{xx} F_1^2 + 2\sigma_{xy} F_1 G_1 + \sigma_{yy} G_1^2 \\ \qquad + \sigma_x (F_{1x} F_1 + F_{1y} G_1) + \sigma_y (G_{1x} F_1 + G_{1y} G_1) \Big|_{\substack{x=x_0 \\ y=y_0}} \end{cases}$$

Then, easily we obtain the following criterion due to J. André and P. Seibert.

**CRITERION**: *For the point $P(x_0, y_0)$ on the line $H_1$,*

1° *if $\sigma_1'(0) < 0$, then the case $(A)$ occurs,*

2° *if $\sigma_1'(0) > 0$, then the case $(B)$ occurs,*

3° *when $\sigma_1'(0) = 0$, if $\sigma_1''(0) > 0$ (or $< 0$), then the case* (C) *(or the case $(D)$) occurs.*

Analogously, the same fact is considered for the equations

$$\begin{cases} \dot{x} = X(x, y, \varphi_2(\sigma(x, y))) \equiv F_2(x, y) \\ \dot{y} = Y(x, y, \varphi_2(\sigma(x, y))) \equiv G_2(x, y), \end{cases}$$

which are defined on the sheets $S_2$, with respect to the same switching line $H_1$. But in this time, since the line $H_1$ is not a boundary for the sheet $S_2$, we need not extend the region $S_2$ as in the case for the sheet $S_1$.

On the switching line $H_2$ the same holds too. But in this case, the situation of the relation of $S_1$ with $S_2$ is in the opposite position. Therefore, we are convenient to consider that the cases $(A)$ and $(B)$, $(C)$ and $(D)$ are alternated respectively.

## § 4. Classification of the points on the switching lines.

In this section, we investigate the behavior of the paths in the neighborhood of the switching lines $H_1$ and $H_2$, when the two sheets $S_1$ and $S_2$ are combined with the lines $H_1$ and $H_2$ in the way as stated in § 3. In the following, we shall classify the possibilities of occurence of the aspects of paths by combinations of the cases on $S_1$ with the cases on $S_2$. By combination of cases, there yield sixteen cases for each of $H_1$ and $H_2$, but we can take up five cases as typical ones by classification under the principle that how many paths passing through the point on the switching lines $H_1$ and $H_2$ there exist into the future, and how many paths there exist into the past.

The notation, say, $(A)-(B)$ denotes that in the region $S_1$ there occurs the case $(A)$, and the case $(B)$ occurs in the region $S_2$ with respect to $H_1$. (For $H_2$, the same thing goes on satisfactorily by alternation of $S_1$ and $S_2$, $(A)$ and $(B)$, and $(C)$ and $(D)$.)

$1°$ : $(A)-(A)$. In this case, there exists the unique path $\gamma_2^+$ passing through the point $P \in H_1$ into the future, but there are two paths $\gamma_1^-$ and $\gamma_2^-$ into the past. The path $\gamma_1^-$ on $S_1$ is continued to the path $\gamma_2^+$, which is contained in the $(-)-$side of $S_2$, after the touching of the point $P$ as $t$ increases. We call such a point a transition point (Fig. 9). The case $(A)-(D)$ belongs to this class.

$2°$ : $(A)-(B)$. In this case, there exists the unique path $\gamma_2^+$ passing through the point $P$ into the future, but there are two paths $\gamma_1^-$ and $\gamma_2^-$ into the past tco. The path $\gamma_1^-$ on $S_1$ is continued to the path $\gamma_2^+$, which is contained in the $(+)-$side of $S_2$, after the touching of the point $P$ as $t$ increases. We call such a point a reflecting point (Fig. 10). The case $(A)-(C)$ belongs to this class.

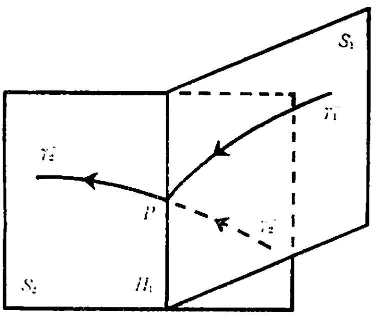

Fig. 9.

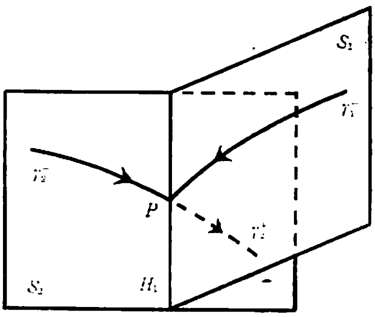


Fig. 10.

$3°$ : $(B)-(A)$. In fhis case, these exist two paths $\gamma_1^+$ and $\gamma_2^+$ passing through the point $P$ into the future, but there is only one path $\gamma_2^-$ into the past which is contained in $S_2$. But the path $\gamma_2^-$ only passes across $H_1$

remaining to stay in $S_2$, since any path does not enter from $S_2$ to $S_1$ traversing $H_1$. We call such a point a maintaining point (Fig. 11). The cases $(B)-(B)$, $(B)-(C)$ and $(B)-(D)$ belong to this class.

4° : $(C)-(A)$. In this case, there exist two paths $\gamma_1^+$ and $\gamma_2^+$ passing through the point $P$ into the future and there also exist two paths $\gamma_1^-$ and $\gamma_2^-$ into the past. We call such a point an ambiguous point (Fig. 12). The cases $(C)-(B)$, $(C)-(C)$ and $(C)-(D)$ belong to this class.

5° : $(D)-(A)$. In this case, there exists the unique path $\gamma_2^+$ passing through the point $P$ into the future and there exists the unique path $\gamma_2^-$ into the past. We call such a point a degeneration point. The cases $(D)-(B)$, $(D)-(C)$ and $(D)-(D)$ belong to this class.

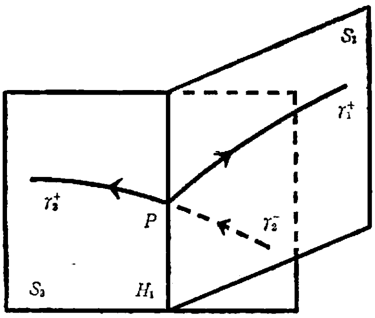

Fig. 11.

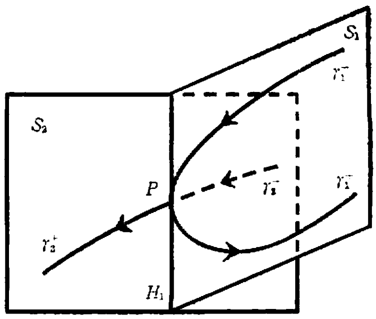

Fig. 12.

In conclussion, we have:

"*For the differential equations* (2.1) *under the assumptions above stated, the existence of solutions always guaranteed. Moreover, if there is no any ambiguous point on the two switching lines, then for the path starting from a point which is off the switching lines, the uniqueness of the solution into the future is also guaranteed*".

Remark 1. In general, the continuity property on the initial point of the path does not hold. It is easily convinced by considering the behavior of paths in the neighborhood of the ambiguous point.

Remark 2. When we need not recognize the distinction between transition points and reflecting points, we may call the point of these classes the confluent point. On the contrary, we may call the maintaining point the apparently bifurcation point. Moreover, we may call the ambiguous point the complex point by considering as the compound of the above.

## § 5. The nonadmissible cases.

When the system is not admissible, we can find various aspect even for the elementary critical points. In the following we pick up some typical cases. Although we do not state the definitions in detail, all of them are self-explaining.

1° When the point on $H_1$ is a stable node for the differential equations defined on the extended region $\tilde{S}_1$ and is an ordinary point for the equations on $S_2$, then the point is a confluent point, that is, the infinitly many paths gather in this point and only one path goes out of this point.

2° When the point on $H_1$ is an unstable node for the equations on $\tilde{S}_1$ and is an ordinary point of the equations on $S_2$, then the point is an apparently bifurcation point, that is, there is one path approaching to this point and there are infinitly many paths going out of this point. But the path approaching to this point is not continued to the paths going out of this point except the path staying on $S_2$.

3° When the point on $H_1$ is a saddle point for the equations on $\tilde{S}_1$ and at least two separatorices of it are contained in $S_1-H_1$, then the point is a complex point.

4° When the point on $H_1$ is a center for the equations on $\tilde{S}_1$ and is an ordinary point for the equations on $S_2$, then the point is a degeneration point.

5° When the point on $H_1$ is a stable node for the equations on $S_2$ and is a point of the case (A) on $S_1$, then the point is an end point, that is, all of the paths approach to this point but they cease continuing into the future after touching.

6° When the point on $H_1$ is a stable node for the equations on $S_2$ and is a point of the case $(B)$ on $S_1$, then the point is an apparently confluent point.

7° When the point on $H_1$ is an unstable node for the equations on $S_2$ and is a point of the case $(A)$ on $S_1$, then the point is a bifurcation point.

8° When the point on $H_1$ is an unstable node for the equations on $S_2$ and is a point of the case $(B)$ on $S_1$, then the point is a starting point, that is, all of paths start from this point.

9° When the point on $H_1$ is a center for the equations on $S_1$ and is also a center for the equations on $S_2$, then the point is a stagnation point, that is, the path consists of only one point.

When the switching line $H_1$ is coincident with a part of the path of the equations on $S_1$ or $S_2$, we can take up two typical cases.

1° When the part of the path on $S_1$ is coincident with $H_1$ then the point is an ambigous point.

2° When the part of the path on $S_2$ is coincident with $H_1$ then the point is considered as the congruent case of the reflecting point and transition point, therefore, it is confluent point.

## § 6. Chattering zone for the special case.

Now, we consider the case where we can construct the region $G$ contained in $R$ overlapped by $S_1$ and $S_2$ as follows: The boundary of the region $G$ consists of four arcs, namely, two subarcs of the switching lines $H_1$ and $H_2$ and two certain arcs as in Fig. 13. The points on the arcs $AD$ and $BC$ are reflecting points and all paths passing through the point on the other two arcs $AB$ and $CD$ enter into the region $G$. Moreover, in this region $G$. there no exists any critical point. In this time, if the path once enters into this region $G$, then it cannot go out of this region $G$ and the path repeat to go and return between two switching lines $H_1$ and $H_2$. The path starting from a point $P$ on the arc $AD$ including both endpoints again return to some point $AD$ after being reflected by the arc $BC$. Therefore, we have a continuous point-transformation from $AD$ to $AD$. Since the arc $AD$ is homeomorphic to the segment [0, 1], hence, from Brower's fixed point theorem, we conclude that, if we can find such a region $G$— called a chattering zone —, then there exists an oscillatory motion, namely, a periodic solution (In practical problem, this oscillation, which generally has very high frequency, is well known as the chattering in relays). Especially, in the following special case, we can discuss to construct the chattcring zone in a concrete form.

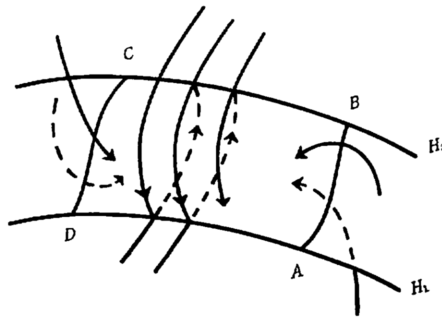


Fig. 13.

Now, we consider the following equations

$$(6.1)\qquad \begin{cases} \dot{x}=a_1x+b_1y+p\varphi(\sigma), \\ \dot{y}=a_1x+b_2y+q\varphi(\sigma), \\ \sigma=cx+dy, \end{cases}$$

where $a_i$, $b_i$ $(i=1,2)$, $p, q, c$ and $d$ are constants and $\varphi(\sigma)$ is a symmetric relay-hysteresis function with the half-width of hysteresis $\delta>0$ and the output of two constant states $\pm 1$ (Fig. 14).

In this time, on the sheet $S_1: \sigma(x, y)=cx+dy \geqq -\delta$, the equations of (6.1) are equivalent to

$$(6.2)\qquad \begin{cases} \dot{x}=a_1x+b_1y+p, \\ \dot{y}=a_2x+b_2y+q, \end{cases}$$

and on the sheet $S_2: cx+dy \leqq \delta$, (6.1) is equivalent to

$$(6.3)\qquad \begin{cases} \dot{x}=a_1x+b_1y-p, \\ \dot{y}=a_2x+b_2y-q. \end{cases}$$

If

$$(6.4)\qquad c\begin{vmatrix} b_1 & p \\ b_2 & q \end{vmatrix}+d\begin{vmatrix} p & a_1 \\ q & a_2 \end{vmatrix}\neq\pm\delta\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

then, the system (6.1) is admissible.

Indeed, if, on the switching line $H_1: cx+dy=\delta$ or $H_2: cx+dy=-\delta$, there is a critical point $P(x_0, y_0)$ of the equations of (6.2) or (6.3), then it must be that the determinant of the coeffiecients of the following equations

$$\begin{cases} a_1x_0+b_1y_0\pm p=0, \\ a_2x_0+b_2y_0\pm q=0, \\ cx_0+dy_0\pm\delta=0 \end{cases}$$

vanishes.

Moreover, if the part of path of (6.2) or (6.3) is coincident with the switching line $H_1$ or $H_2$, then, when $d\neq 0$, for some interval $I$ of $x$, the function

$$(6.5)\qquad y=\pm\frac{\delta}{d}-\frac{c}{d}x \qquad \text{for } x\in I$$

is a solution of the following differential equation

$$(6.6)\qquad \frac{dy}{dx}=\frac{a_2x+b_2y\pm q}{a_1x+b_1y\pm p}.$$

Therefore, substituting (6.5) into (6.6), we have

$$\frac{(da_2-cb_2)x\pm dq\pm\delta b_2}{(da_1-cb_1)x\pm dp\pm\delta b_1}=\text{const.} \qquad \text{for } x\in I,$$

hence, the determinant of the coefficients must vanish. This proves (6.4).

Put

$$\begin{aligned} s(x, y)&=c(a_1x+b_1y)+d(a_2x+b_2y) \\ &=(ca_1+da_2)x+(cb_1+db_2)y. \end{aligned}$$

If

$$\Delta=\begin{vmatrix} ca_1+da_2 & cb_1+db_2 \\ c & d \end{vmatrix}\neq 0,$$

then the switching line $H_1$ and straight line $L_1: s(x, y)=-(cp+dp)$ traverse each other. We denote the point of intersection of $H_1$ with $L_1$ by $A$. Since the switching line $H_2$ is parallel to $H_1$, $L_1$ intersects with $H_2$ at the point $B$.

Similarly we consider a straight line $L_2: s(x, y)=cp+dq$ and we denote the points of intersection of the line $L_2$ with two lines $H_1$ and $H_2$ by $D$ and $C$ respectively. Then we obtain a parallelogram $ABCD$ (Fig. 15).

Now, we investigate the directions of paths on each side of the parallelogram $ABCD$.

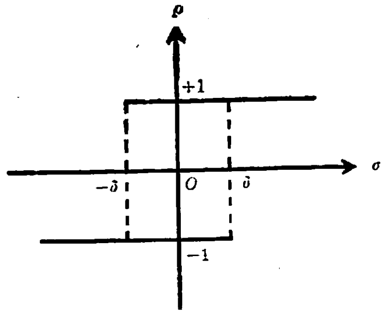


Fig. 14.

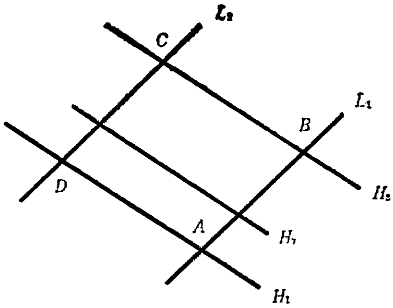


Fig. 15.

In order that any point on $AD$ is a reflecting point, it is sufficient that for the path defined by (6.2) $\sigma_1'<0$, and $\sigma_2'>0$ for the path defined by (6.3) where

$$\sigma_i'=\sigma_i'(0)=\frac{d\sigma}{dt}(x_i(t, x, y), y_i(t, x, y))\Big|_{t=0} \qquad (i=1, 2)$$

for $(x, y)\in H_1$.

Therefore, for the paths of (6.2), it must be

$$c(a_1x+b_1y+p)+d(a_2x+b_2y+q)<0,$$

hence, we have

$$(6.7) \qquad s(x, y)=(ca_1+da_2)x+(cb_1+db_2)y<-(cp+dp).$$

Similarly, for the paths of (6.3), we have

$$(6.8) \qquad s(x, y)=(ca_1+da_2)x+(cb_1+db_2)y>(cp+dq).$$

If $cp+dq<0$, then the segment of $H_1$ given by both inequalities (6.7) and (6.8) has nonempty intersection. In the sequel, we have that if $cp+dq<0$, then the point on $AD$ is reflecting point, The same discussion is valid for the side $BC$ and we obtain the same inequality.

Let $H_\gamma$ be the straight line:

$$H_\gamma: \sigma(x, y)=cx+dy=\gamma \qquad (-\delta<\gamma<\delta),$$

then the coordinates of the point of intersection of $H_\gamma$ with $L_1$ are

$$(6.9) \qquad \begin{cases} x=-\dfrac{1}{\Delta}\{\gamma(cb_1+db_2)+d(cp+dq)\}, \\ y=\dfrac{1}{\Delta}\{\gamma(ca_1+da_2)+c(cp+dq)\}. \end{cases}$$

Firstly, from the conditon posed on the segment $AB$ and negaitiveness of $cp+dq$, it must be that, along the path of (6.2), $s=s(x,y)$ is decreasing at $(x,y)\in AB$. Hence,

(6.10) $$s'=(ca_1+da_2)(a_1x+b_1y+p)+(cb_1+db_2)(a_2x+b_2y+q)<0.$$

Substituting (6.9) into (6.10) and rearranging, we have the following inequality:

(6.11) $$U\gamma+V<W \qquad (-\delta<\gamma<\delta)$$

where

$$\begin{cases} U=a_1b_2-a_2b_1, \\ V=-(a_1+b_2)(cp+dq), \\ W=-\{(ca_1+da_2)p+(cb_1+db_2)q\} \end{cases}$$

Similarly, from the condition posed by path of (6.3), we have

(6.12) $$U\gamma+V<-W \qquad (-\delta<\gamma<\delta).$$

The same discussion is valid for the side $CD$ and we obtain the same inequalities as (6.11) and (6.12).

Moreover, the parallelogram $ABCD$ does not contain any critical point of the equations of (6.2) or (6.3). Obviously, the critical point lay itself on the lines $L_1$ or $L_2$. Since, on the segments $AB$ and $CD$, the direction of path is exactly determined, there no exists any critical point on the segment $AB$ and $CD$.

And we find that the condition (6.11) iucludes the condition (6.4).

Summarizing the above, we have:

"*For the admissible system* (6.1), *when* $\Delta\neq 0$, *the parallelogram* $ABCD$ *can be coustructed and moreover, if* $cp+dq<0$ *and* $|U|\delta+V<-|W|$, *then the parallelogram* $ABCD$ *is a chattering zone.*"

## § 7. Example.

Now, we consider the following equations

(7.1) $$\begin{cases} \dot{x}=-kx+\varphi(\sigma), \\ \dot{\sigma}=cx-\rho\varphi(\sigma) \end{cases}$$

where $k$, $c$ and $\rho$ are positive constants and $\varphi(\sigma)$ is a symmetric relay-hysteresis function with the half-width of hysteresis $\delta>0$ and the output of two states $\pm M(M>0)$.

Put

$$\begin{cases} S_1=\{(x,\sigma);\sigma\geqq-\delta\}, \\ S_2=\{(x,\sigma);\sigma\leqq\delta\}. \end{cases}$$

In the sheet $S_1$, the equations of (7.1) are equivalent to

$$(7.2)\quad \begin{cases} \dot{x}=-kx+M, \\ \dot{\sigma}=cx-\rho M. \end{cases}$$

Since these equations are linear equations with constant coefficients, we easily obtain explicitly their solution which passes through the point $P(x_0, -\delta)$ on the line $H_1 : \sigma=-\delta$ at the time $t=0$. Namely, we have

$$\begin{cases} x_1(t)=\frac{M}{k}(1-e^{-kt})+x_0e^{-kt}, \\ \sigma_1(t)=\frac{c}{k}(x_0-\frac{M}{k})(1-e^{-kt})+M(\frac{c}{k}-\rho)t-\delta, \end{cases}$$

as long as $(x_1, \sigma_1)\in S_1$.

Similarly in the sheet $S_2$, the equations of (7.1) are equivalent to

$$(7.3)\quad \begin{cases} \dot{x}=-kx-M, \\ \dot{\sigma}=cx+\rho M. \end{cases}$$

And we obtain the solution of (7.3) such that the path passes through the point $P(x_0, \delta)$ on the line $H_2 : \sigma=\delta$ at the time $t=0$. Namely, we have:

$$\begin{cases} x_2(t)=\frac{M}{k}(e^{-kt}-1)+x_0e^{-kt}, \\ \sigma_2(t)=\frac{c}{k}(x_0+\frac{M}{k})(1-e^{-kt})-M(\frac{c}{k}-\rho)t+\delta, \end{cases}$$

as long as $(x_2, \sigma_2)\in S_2$.

Therore, if $\rho k>c$, then the aspect of paths on each of the sheets $S_1$ and $S_2$ is as in Fig. 16.

In this case, it is easily seen that this system (7.1) is admissible if $\rho k\neq c$, and, by checking the criterion on the chattering zone stated in the preceding section, or as is easily seen intuitively from Fig. 16, we find that the parallelogram $ABCD$ is a chattering zone. Therefore, there exists a periodic motion in $ABCD$.

On the other hand, if $\rho k<c$, then we can similarly draw the aspect of paths. But in this case, there is only apparent chattering zone, of which all of paths eventually go out.

If the half-width of hysteresis $\delta$ tends to zero, the chattering zone clearly degenerates to a rest interval (=a set of end points). Therefore, the existence property into the future is broken off by the limiting process and there yields some gap between the case; $\delta>0$ and the limiting case $\delta=0$. But if we interpret the meaning of solutions for the case $\delta=0$ by the definition due to А. Ф. Филиппов [3], then we can supply this gap to a certain extent.

In this case, our problems reduce to the problems of differential equa-

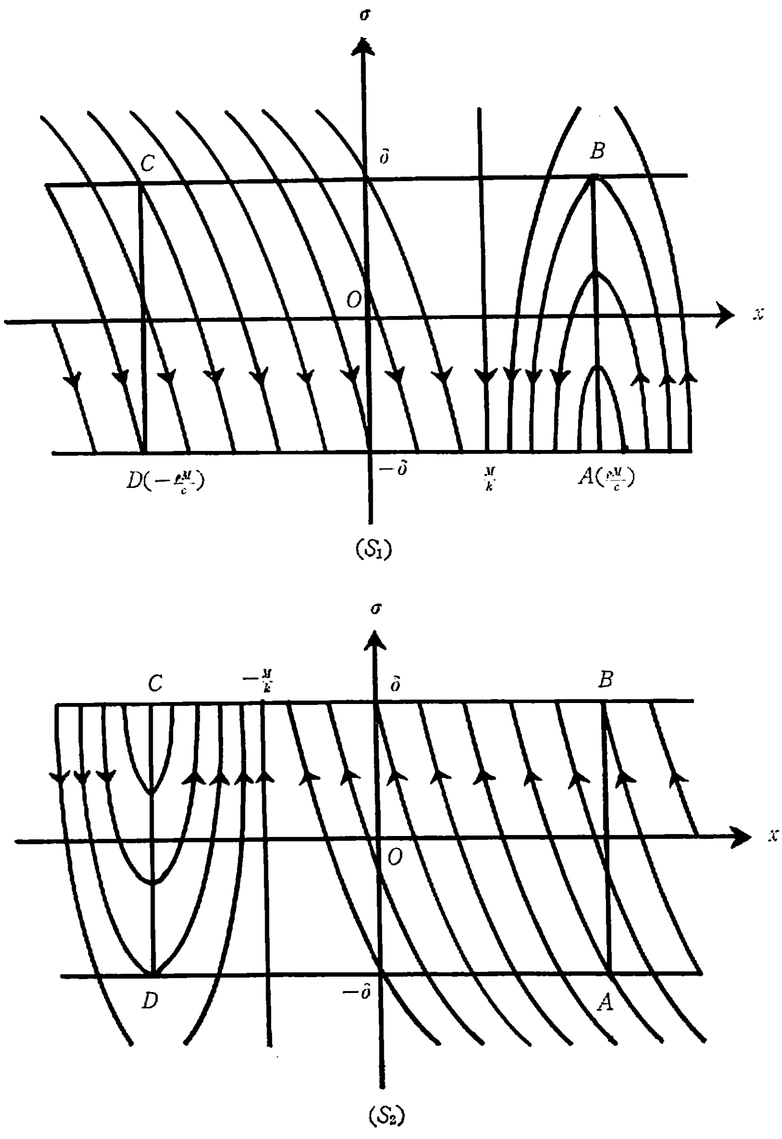


Fig. 16.

tions with discontinuous right members, and various facts are known (see S. Lefschetz [4]).

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