A NOTE ON UNIFORM INTEGRABILITY

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1. Let Ω be a measure space with a measure μ. It is known that in $L_1(\Omega, \mu)$ = the space of integrable functions on Ω with respect to μ, the following conditions for a subset $\mathfrak{U}$ of L_1 are equivalent to each others, provided that the measure μ is finite on Ω.

(1) $\mathfrak{U}$ is uniformly integrable;

(2) $\mathfrak{U}$ is bounded (in the sense of L_1-norm) and for $\mu(e_i) \downarrow 0$,

$\int_{e_i} f \, d\mu \to 0$ (as $i \to \infty$) uniformly on $f \in \mathfrak{U}$;

(3) $\mathfrak{U}$ is relative weakly compact ;

(4) $\mathfrak{U}$ is relative weakly sequential compact.

Here, weak topology on L_1 means the weak topology by L_∞ ; this topology is denoted by $\sigma(L_\infty)$.

In this note, we shall generalize the notion of uniform integrability in the case where μ is not necessary finite. This situation is also applicable to the cases of measurable function spaces and more general complete vector lattices.

For simplicity, we shall discuss mainly the case of $L_1(\Omega, \mu)$. The equivalence of (2) and (3) is fully discussed by Nakano and Amemiya ([1], [4]) in the case of complete vector lattice. (L_1 is an example of complete vector lattice). In this case, the condition (2) is reformed in the statement of vector lattice [$\mathfrak{U}$ is equi-continuous in Nakano's terminology].

Even though (3) and (4) are equivalent in the case of L_1-spaces, in general measurable function spaces this equivalence does not follow.

2. We shall consider the case where μ is not necessary finite. Let $\Omega = \bigcup_\alpha S_\alpha$ where S_α is pairwise disjoint μ-measurable set in Ω and μ is σ-finite measure on S_α for each α and every finite measurable set of Ω is contained in countable union of S_α.

For $f \in L_1(\mu)$, we denote by $[f]$ an operator defined as follows :

$$[f]g = \bigcup_{n=1}^{\infty} (g \frown n|f|) \text{ for } g \geq 0 \text{ and } g \in L_1(\mu)$$

and

$$[f]g = [f]g^+ - [f]g^- \text{ for arbitrary } g \in L_1(\mu).$$

It is easy to see that $[f]$ is a linear operator on $L_1(\mu)$ with $[f] = [|f|]$ and idempotent. If $\chi_{[f]}$, is a characteristic function on a subset $\{x ; f(x) \neq 0, x \in \Omega\}$,

$$[f]\, g = \chi_{[f]} \cdot g \text{ for every } g \in L_1(\mu).$$

A subset $\mathfrak{U} \subset L_1(\mu)$ is called *uniformly integrable with respect to* $f \varepsilon L_1(\mu)$, if

$$A_n = \sup_{g \varepsilon \mathfrak{U}} \int_{|g| \geqq n|f|} [f] g \, d\mu \to 0 \text{ as } n \to +\infty.$$

Theorem 1 *Let* $\mathfrak{U}$ *be uniformly integrable with respect to* $f \varepsilon L_1(\mu)$, *and* $P_1 \supset P_2 \supset P_3 \supset \cdots$ *with* $\mu(\bigcap_{i=1}^{\infty} P_i) = 0$, *then* $\sup_{g \varepsilon \mathfrak{U}} \int_{P_n} [f] |g| d\mu \to 0$ *as* $n \to \infty$ *and* $[f]\mathfrak{U}$ *is bounded.* (*in the sense of* L_1-*norm*)

Proof. For every $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$A_n = \sup_{g \varepsilon \mathfrak{U}} \int_{|g| \geqq n|f|} [f] |g| d\mu \leqq \varepsilon \quad \text{for} \quad n \geqq N.$$

Since $P_1 \supset \cdots \supset P_n \cdots$ with $\mu(\bigcap_{i=1}^{\infty} P_i) = 0$, we have

$$\int_{P_n} N|f| \, d\mu \to 0 \text{ as } n \to \infty \qquad \text{for a fixed number } N.$$

Hence $\int_{P_n} [f] |g| \, d\mu \leqq \int_{P_n} N|f| \, d\mu + \int_{|g| \geqq N|f|} [f] |g| d\mu \leqq 2\varepsilon$

for sufficient large n and for every $g \varepsilon \mathfrak{U}$.

It is easy to see that $[f]\mathfrak{U}$ is norm bounded.

Theorem 2 *Let* $\mathfrak{U}$ *be a subset of* $L_1(\mu)$ *such that* $\sup_{g \varepsilon \mathfrak{U}} \int_{P_n} [f] \, |g| d\mu \to 0$ *for* $P_1 \supset P_2 \supset \cdots \supset P_n \supset \cdots$ *with* $\mu(\bigcap_{n=1}^{\infty} P_n \cap \{x; f(x) \neq 0\}) = 0$ *and* $[f]\,\mathfrak{U}$ *is norm bounded. Then* $\mathfrak{U}$ *is uniformly integrable with respect to* f.

Proof. If $\mathfrak{U}$ is not uniformly integrable with respect to f, then there exists $\varepsilon > 0$ such that

$$A_n = \sup_{g \varepsilon \mathfrak{U}} \int_{|g| \geqq n|f|} [f] |g| d\mu \quad \geqq \varepsilon > 0$$

for $n = 1, 2 \cdots$.

Since $[f]\,\mathfrak{U}$ is norm bounded, there exists a number M and $g_n \varepsilon \mathfrak{U}$ with $M \geqq \int_{|g_n| \geqq n|f|} [f] |g_n| d\mu \geqq \frac{\varepsilon}{2} (n = 1, 2, \cdots)$ i.e.

$$M \geqq \int_{P_n} n|f| \, d\mu = n \int_{P_n} |f| d\mu$$

where $P_n = \{x; |g_n(x)| \geqq n \, |f(x)|\}$. Hence $\int_{P_n} |f| \, d\mu \to 0$ as $n \to \infty$ and $\sup_{g \varepsilon \mathfrak{U}} \int_{Q_n} [f] |g| d\mu \geqq \frac{\varepsilon}{2}$ for a suitable $Q_n \subset \{x; f(x) \neq 0\}$ decreasing with $\mu(\bigcap_{n=1}^{\infty} Q_n) = 0$; it means that $\mathfrak{U}$ does not satisfy the assumption of Theorem 2.

Theorem 3 *If* $[f_1] = [f_2]$, *and* $\mathfrak{U}$ *is uinformly integrable with respect to* f_1, *then* $\mathfrak{U}$ *is uniformly integrable with respect to* f_2.

Proof. Since $[f_1]=[f_2]$ is equivalent to $\{x: f_1(x)\neq 0\}=\{x; f_2(x)\neq 0\}$ except μ-measure zero set, Theorem 3 is a direct consequence of Theorem 1 and 2.

Theorem 4. *If* $\mathfrak{U}$ *is uniformiy integrable with respect to every* $f\in L_1(\mu)$, *then* $\mathfrak{U}$ *is norm-bounded.*

Proof. Suppose that $\mathfrak{U}$ is not norm bounded. Then there exists a sequence of $g_n \varepsilon \mathfrak{U}$ $(n=1, 2, \cdots)$ such that $\int|g_n|\,d\mu\geqq n$. Since $S_\alpha\cap\{x; g_n(x)\neq 0\}\neq\phi$ only for countable α for each n, there exists at most countable S_{α_i}, for which $S_{\alpha_i}\cap\{x; g_n(x)\neq 0\}\neq\phi$ for some n $(i=1, 2, \cdots)$. Since μ is σ-finite on S_{α_i}, there exists a function $f\varepsilon L_1(\mu)$, whose support is exactly $\bigcup_{i=1}^{\infty}S_{\alpha_i}$. Because $\mathfrak{U}\supset\{g_n\}$ is uniformly integrable with respect to f, $[f]\mathfrak{U}\supset\{g_n\}$ is norm bounded ; this is a contradiction.
It is easy to see :

Theorem 5. *If* $\mathfrak{U}$ *is norm bounded and* $\int_{P_n}|g|\,d\mu\to 0$ *uniform on* $g\in\mathfrak{U}$ *for* $P_1\supset P_2\supset\cdots\supset P_n\supset\cdots$ *with* $\mu(\bigcap_{n=1}^{\infty}P_n)=0$, *then* $\mathfrak{U}$ *is uniformly integrable with respect to all* $f\ \varepsilon L_1(\mu)$.
We shall say that $\mathfrak{U}$ is *uniformly integrable* if $\mathfrak{U}$ is uniformly integrable with respect to all $f\in L_1$.

Theorem 6. *If* $\mathfrak{U}$ *has the property*: $\int_{P_n}|g|\,d\mu\to 0$ *uniform on* $g\in\mathfrak{U}$ *for* $P_1\supset P_2\supset\cdots P_n\supset\cdots$ *with* $\mu(\bigcap_{n=1}^{\infty}P_n)=0$, *and* Ω *is non-atomic with respect to* μ, *then* $\mathfrak{U}$ *is norm bounded.*

Proof. Suppose that $\mathfrak{U}$ is not norm bounded. There exists a sequence of elements $g_n\in\mathfrak{U}$ with $\int|g_n|\ d\mu\geqq 2n$. We can find a μ-measurable finite set P_n with $\int_{P_n}|g_n|\,d\mu\geqq n$. Put $Q_n=P_1\cup P_2\cup\cdots\cup P_n$ and $Q_0=\bigcup_{n=1}^{\infty}Q_n$. Then, Q_0-Q_n is decreasing with $\bigcap_{n=1}^{\infty}(Q_0-Q_n)=\phi$. Hence, observing $\int|g|\,d\mu\to 0$ as $n\to\infty$, there exists $Q=Q_m$ with $\int_Q|g_n|\,d\mu\geqq n/2$. Since $\mu(Q)<+\infty$ and Ω is non-atomic, we can decompose Q into $Q_{1,1}$ and $Q_{1,2}$ with $Q=Q_{1,1}\cup Q_{1,2}$; $Q_{1,1}\cap Q_{1,2}=\phi$; $\mu(Q_{1,1})=\frac{\mu(Q)}{2}=\mu(Q_{1,2})$. We can select $R_1=Q_{1,1}$ or $Q_{1,2}$ such that $\int_{R_1}|g_n|\,d\mu$ $(n=1, 2, \cdots)$ is not bounded. By the same way and by induction, we can find $R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$ with $\mu(R_n) = \frac{\mu(Q)}{2^n}$ and for all m $\int_{R_m} |g_n| d\mu$ $(n=1, 2, \cdots)$ is not bounded. But $\mu(\bigcap_{m=1}^{\infty} Q_m) = 0$, and so this is a contradiction.

Therem 7. *If μ is non-atomic on $[f] L_1(\mu)$, then $\mathfrak{U}$ is uniform integrable with respect to f if and only if $[f]$ $\mathfrak{U}$ has the property* of *Theorem* 6.

Normal hull of $\mathfrak{U}$ is a subset of all f, to which there exists g with $|f| \leqq |g|$ for some $g \in \mathfrak{U}$.

Theorem 8. *If* $\mathfrak{U}$ *is uniformly integrable with respect to all $f \varepsilon L_1(\mu)$, then $\bigcup_{g \varepsilon \mathfrak{U}} \{x ; g(x) \neq 0\}$ is contained in at most countable union of S_α. Hence, there exists $f_0 \in L_1(\mu)$ such that $[f_0]\mathfrak{U} = \mathfrak{U}$.*

Proof. If $\mathfrak{U}$ is uniformly integrable with respect to f, then the normal hull $\tilde{\mathfrak{U}}$ of $\mathfrak{U}$ is uniformly integrable with respect to f. Suppose that $\bigcup_{g \varepsilon \mathfrak{U}} \{x ; g(x) \neq 0\}$ is not contained in any countable union of S_α. Then, we can find an element $f_n \in L_1(\mu)$, whose supports are pairwise disjoint and $\|f_n\| = \int |f_n| d\mu \geqq \varepsilon > 0$ for some number ε for every integer n. Considering $g_n = \frac{1}{2^n \|f_n\|} f_n$ and $f = \sum_{n=1}^{\infty} g_n$, we see $f \varepsilon L_1(\mu)$ and $\{g_n\}$ is not uniformly integrable with respect to f. Hence, $\tilde{\mathfrak{U}}$ and $\mathfrak{U}$ are not uniformly integrable with respect to f. This is a contradiction.

Theerem 9. $\mathfrak{U}$ *is relative weakly compact if and only if* $\mathfrak{U}$ *is uniformly integrable with respect to all $f \varepsilon L_1(\mu)$.*

Proof. Let $\mathfrak{U}$ be uniformly integrable. Then the normal hull $\tilde{\mathfrak{U}}$ of $\mathfrak{U}$ is also uniformly integrable. We define then the semi-norm in L_∞ such that

$$\|g\|_{\mathfrak{U}} = \sup_{f \varepsilon \tilde{\mathfrak{U}}} |\langle f, g \rangle| = \sup_{f \varepsilon \tilde{\mathfrak{U}}} \left| \int f . g \, d\mu \right|.$$

(*) For every $P_n \supset P_{n+1} \supset \cdots$ with $\mu(\bigcap_{n=1}^{\infty} P_n) = 0$, we have $\|\chi_{P_n} . g\|_{\mathfrak{U}} \to 0$.

We have also $\|\chi_{P_\lambda} . g\|_{\mathfrak{U}} \to 0$ by Theorem 8 for any directed system P_λ with $\mu(\bigcap P_\lambda) = 0$.

τ is a linear topology defined by all semi-norm $\|\cdot\|_{\mathfrak{U}}$ where $\mathfrak{U}$ is uniformly integrable subset of $L_1(\mu)$.

Any τ-continuous linear functional on L_∞ is considered as an element of $L_1(\mu)$ by (*) and conversely any $f \varepsilon L_1(\mu)$ is a τ-continuous linear functional on $L_\infty(\mu)$ i. e. $L'_\infty[\tau] = L_1$. Hence, every uniformly integrable subset $\mathfrak{U}$ of L_1 is equi-contiunous by τ. By Alaoglu-Bourbaki's theorem, $\mathfrak{U}$ is a relative weakly compact set.

Conversely, if $\mathfrak{U}$ is not uniformly integrable, then we can find a sequence of measurable sets $P_1 \supset P_2 \supset \cdots$ with $\mu(\bigcap_{n=1}^{\infty} P_n)=0$ and f_n such that

$$\int_{P_n} |f_n| \, d\mu \geqq \varepsilon.$$

Suppose that $\mathfrak{U}$ is relative weakly compact. By Smulian's theorem and Eberlein's theorem $\mathfrak{U}$ is relative weakly sequentially compact. There exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that $f_{n_i} \to f$ (weak). By Nikodym's theorem, the set functions $\lambda_i(P_n)=\int_{P_n} f_{n_i} \, d\mu$ tends to 0 uniformly on $i=1, 2, \ldots$. This is a contradiction.

Theorem 10. *Let μ be a finite measure on Ω. A subset $\mathfrak{U}$ of $L_1(\mu, \Omega)$ is uniformly integrable if and only if there exists a convex function $\phi(t) \geqq 0$ for $t \geqq 0$ with $\lim_{t\to\infty} \frac{\phi(t)}{t} = +\infty$ such that $\sup_{f\in\mathfrak{U}} \int \phi(|f|) d\mu < +\infty$.*

Even if this theorem is famous and easy, I can not find a detail proof of this theorem. We shall skech a proof of this theorem.

Proof. Let $\mathfrak{U}$ be uniformly integrable. For every integer $i \geqq 1$, there exists $n_i \uparrow_i \infty$ with $\sup_{f\in\mathfrak{U}} \int_{|f|\geqq n_i} |f| \, d\mu \leqq \frac{1}{2^i}$. At first, for $i=1$, we choose a point $(n_1, \sqrt{2}\, n_1)=(m_1, \sqrt{2} m_1)$ in the plane. Next for $i \geqq 2$ we find $\frac{(\sqrt{2})^i n_i - \sqrt{2}\, n_1}{n_i - n_1} \geqq (\sqrt{2})$. Hence, $\min_{i\geqq 2} \frac{(\sqrt{2})^i n_i - \sqrt{2}\, n_1}{n_i - n_1}$ exists and we put $m_2 = n_i$ and $i=i_2$ such that for n_i, $\frac{(\sqrt{2})^i n_i - \sqrt{2}\, m_1}{n_i - m_1}$ takes the minimum value. For $i \geqq i_2$, we find $\frac{(\sqrt{2})^i n_i - (\sqrt{2})^{i_2} m_2}{n_i - m_2} \geqq (\sqrt{2})^{i_2}$ and put $m_3 = n_i$ and $i=i_3$ such that for $n_i = m_3$, $\frac{(\sqrt{2})^i n_i - (\sqrt{2})^{i_2} m_2}{n_i - m_2}$ takes the minimum value. By induction, we find $\frac{(\sqrt{2})^i n_i - (\sqrt{2})^{i_{j-1}} m_{j-1}}{n_i - m_{j-1}}$ takes the minimum value for $m_j = n_i$ and $i=i_j$.

Now, we draw a polygon combining $(0, 0)$, $(m_1, \sqrt{2}\, m_1)$, $(m_2, (\sqrt{2})^{i_2} m_2)$, $\cdots$, $(m_n, (\sqrt{2})^{i_n} m_n)$, $\cdots$. This polygon defines a convex function $\Phi(t)$ for $t \geqq 0$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty$. Let $\Psi(t) = (\sqrt{2})^i n_i$ for $n_{i-1} < t \leqq n_i$, *then* $t \leqq \Phi(t) \leqq \Psi(t)$. For $f \in \mathfrak{U}$, $\int_\Omega \Psi(|f|) \, d\mu \leqq \sum_{i=0}^{\infty} \frac{1}{(\sqrt{2})^i} < +\infty$. Hence we have $\sup_{f\in\mathfrak{U}} \int \Phi(|f|) \, d\mu < +\infty$.

Conversely, if a convex function Φ satisfies $\lim_{t\to\infty}\frac{\Phi(t)}{t}=+\infty$ and $\sup_{f\in \mathfrak{U}}\int \Phi(|f|)\,d\mu=M<+\infty$, then for every $n\geqq 1$, there exists m such that $u(x)\geqq m$ implies $\Phi(|u(x)|)\geqq n|u(x)|$.

$$M\geqq\int_{|u|\geqq m}\Phi(|u|)\,d\mu=n\int_{|u|\geqq m}|u(x)|\,d\mu.$$

i. e. $\frac{M}{n}\geqq\int_{|u|\geqq m}|u(x)|\,d\mu$ for all $u\in\mathfrak{U}$. This means that $\mathfrak{U}$ is uniformly integrable.

We shall consider the case where μ is not finite on Ω. For this, we need another definition of uniformly integrable subsets.

Theorem 11. *A subset* $\mathfrak{U}$ *of* $L_1(\mu)$ *is uniformly integrable with respect to* f, *if and only if there exists a sequence of* $a_n(x)\in L_1(\mu)$ *with* $0\leqq a_n(x)\uparrow_n\infty$ *and* $[a_n]=[f]$ *such that* $\sup_{u\in\mathfrak{U}}\int_{|u|\geqq|a_n|}|u(x)|\,d\mu\to 0$ *as* $n\to\infty$.

Proof. By the same argument of Theorem 1, if $\mathfrak{U}$ satisfies the conditions of the present theorem, $\mathfrak{U}$ is uniformly integrable with respect to f. Let $\mathfrak{U}$ be uniformly integrable with respect to f. Suppose that there exists g_n $(n=1, 2, \cdots)$ with

$$M\geqq\int_{|g_n|\geqq|a_n|}[f]\,|g_n|\,d\mu\geqq\varepsilon>0,$$

because $[f]\mathfrak{U}$ is bounded by Theorem 2.

$$M\geqq\int_{P_n}a_n(x)\,d\mu$$

where $P_n=\{x\in\Omega\,;\ |g_n(x)|\geqq a_n(x)\}\cap\{x\,;\ f(x)\neq 0\}$.
Since $[a_n]=[f]$ and $a_n(x)\uparrow\infty$, and
$\int_{P_n}a_1(x)\,d\mu\to 0$ as $n\to\infty$, we have $\sup_{g\in\mathfrak{U}}\int_{Q_n}[f]g\,d\mu\geqq\frac{\varepsilon}{2}$ for some μ-measurable Q_n with $\mu(\bigcap_{n=1}^{\infty}Q_n)=0$. This is a contradiction.

Now, we have the following theorem.

Theorem 11. $\mathfrak{U}$ *is uniformly integrable if and only if there exists a function* $\Phi(t, x)$ *of two variables* $x\in\Omega$ *and real* $t\geqq 0$ *such that* $\Phi(t, x)$ *is a convex function of* t *for a fixed* $x\in\Omega$, *a measurable function of* x *for a fixed* $t\geqq 0$, *and* $a_n(x)\uparrow\infty$, $\Phi(a_n(x), x)\geqq na_n(x)$ *with* $[a_1]=[a_2]=\cdots=[a_n]=\cdots$, $[a_1]\mathfrak{U}=\mathfrak{U}$ *and*

$$\sup_{f\in\mathfrak{U}}\int\Phi(|f(x)|, x)\,d\mu<+\infty.$$

Proof. Let $\mathfrak{U}$ be uniformly integrable. Then, there exists $0\leqq f\in L_1(\mu)$ with $[f]\mathfrak{U}=\mathfrak{U}$ and $\mathfrak{U}$ is uniformly integrable with respect to f by Theorem

8. By Theorem 3, we can assume that f is a step function whose values may be countably many. For every integer $i=1$, there exists $n_1 < n_2 < \cdots n_i < \cdots$ with

$$\sup_{g \in \mathfrak{U}} \int_{|g| \leq n_i f} |g| \, d\mu \geq \frac{1}{2^i}.$$

By the method discussed in the proof of Theorem 10, we find a convex function $\varphi(t)$ $(t \geqq 0)$ such that for infinitely many i,

$\varphi(n_i) = (\sqrt{2})^i \, n_i$ and $\psi(t) \geqq \varphi(t)$;

where $\psi(t) = (\sqrt{2})^i \; n_i$ for $n_{i-1} < t \leqq n_i$,

We define

$$\varphi(t, x) = f(x) \varphi\left(\frac{t}{f(t)}\right) \qquad \text{for } f(x) \neq 0$$
$$= t^2 \qquad \text{for } f(x) = 0.$$

Then, $\varphi(t, x)$ is a convex function of $t \geqq 0$ for $x \in \Omega$, and measurable function of t for $x \in \Omega$.

$\varphi(n_i f(x), x) = (\sqrt{2})^i n_i f(x)$ for infinitely many of i.

Hence, putting $a_n(x) = n_i f(x)$ for suitable $n_i = n$, then

$\varphi(a_n(x), x) \geqq n a_n(x)$ and $a_n(x) \in L_1(\mu)$ and $[a_n] = [f]$.

We have also,

$$\int \varphi(|g(x)|, x) \, d\mu \leqq \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i < \infty.$$

Conversely, if there exists $\varphi(t, x)$ with the conditions of Theorem for $n \geqq 1$, there exists m such that

$u(x) \geqq a_m(x)$ implies $\varphi(|u(x)|) \geqq n \; u(x)$;

$$M \geqq \int \varphi(|u(x)|, x) d\mu \geqq n \int_{u(x) \geqq a_m} |u(x)| d\mu \qquad \text{for all } u \in \mathfrak{U}.$$

i. e.

$$\frac{M}{n} \geqq \int_{|u| \geqq a_m} |u| d\mu \quad \text{for } u \in \mathfrak{U}.$$

Hence, we see that $\mathfrak{U}$ is uniformly integrable by Theorem 10.

3. The same argument discussed in 2 for $L_1(\mu)$ is also applicable for the case of complete vector lattice which is order dual of some complete vector lattice. For example, $L_1(\mu)$ is order dual of $L_\infty(\mu)$.

A subset $\mathfrak{U}$ of L, which is a complete vector lattice of order dual of a complete vector lattice C is *uniformly integrable with respect to* f if

$$\langle s, [(|g| - nf)^+] g \rangle \to 0 \text{ uniformly on } g \in \mathfrak{U} \quad \text{for every } 0 \leqq s \in C.$$

Then, $\mathfrak{U}$ is uniformly integrable (for all $f \in L$) if and only if $\mathfrak{U}$ is relative weakly compact (i. e. relative compact by (C)). But, essentially the characterization of relative weakly compact sets is obtained by Nakano [4].

If L is a Banach lattice and order dual of a complete vector lattice C and C is conjugate space of L as Banach space, then $\mathfrak{U}$ is relative weakly compact if and only if $\mathfrak{U}$ is relative weakly sequentially compact.

But the equivalence of relative weakly compact sets and relative weakly sequentially compact sets is not true in general.

We shall show that in $L_\infty(\mu)$, there exists $\mathfrak{U}$ which is uniformly integrable (=relative weakly compact), but $\mathfrak{U}$ is not relative weakly sequentially compact.

Let Ω be a set of density of real number space and μ be discrete measure on Ω (i. e. every point measure is one).

We shall consider a countable set $\{f_i\} \subset L_\infty(\mu)$ such that the totality of sequence $f_i(x)(i=1, 2, \cdots)$ $x \in \Omega$ is the totality of the sequence whose coordinate is 1 or -1. For every subsequence $\{f_{i_j}\} \subset \{f_i\}$, we find an element x such that $\{j\ ;\ f_{i_j}(x)=1\}$ and $\{j\ ;\ f_{i_j}(x)=-1\}$ are infinitely many. If g_j is weakly convergent to some g, then $g_j(x) \to g(x)$ for every $x \in \Omega$. Hence the sequence of elements f_{i_j} is not weakly convergent.

The set f_i is relative weakly compact, but not relative weakly sequentially compact. Weak toplogy is here $\sigma(L_1(\mu))$.

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