

# ON $N$ -REGULAR GROUPS OF AUTOMORPHISMS IN SIMPLE RINGS

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In [1], N. Jacobson introduced the notions of  $N$ -groups and reduced orders for groups of automorphisms in rings, and which was used in his Galois theory. Recently, in [6], the present author introduced the notion of reduced indices for  $N$ -groups of automorphisms in division rings, and showed that reduced indices can be used for a characterization of groups of automorphisms. In §1—6 of this paper, this consideration will be extended to simple rings, and moreover, in the study, we shall present some relative Galois correspondences which are of interest, where §1—2 contain notations and preliminary lemmas. In simple rings, the notions of  $N$ -regular groups and regular groups for groups of automorphisms are used in studying groups of automorphisms, which will be defined later (§1), and the regularity of  $N$ -regular groups may be posed as a problem in Galois theory as well as in the theory of centralizers of simple subrings in a simple ring. In the last section 7, as an application, we shall study the regularity of  $N$ -regular groups of automorphisms in simple ring extensions with local finiteness. On these groups, we have a study [3, §2], which will be sharpen in this paper.

**1. Notations.** Throughout the present paper,  $A$  will be a simple ring with minimum condition, and we set  $\mathfrak{A} = \text{Hom}(A, A)$  (acting on the right side of  $A$ ). If  $\mathfrak{C}$  is a subset of  $\mathfrak{A}$ , we denote by  $V_{\mathfrak{A}}(\mathfrak{C})$  the centralizer of  $\mathfrak{C}$  in  $\mathfrak{A}$ , and then  $V_{\mathfrak{A}}(\mathfrak{C}) = \text{Hom}_{\mathfrak{C}}(A, A)$ . For any element  $a$  of  $A$ , we denote by  $a_l$  (resp.  $a_r$ ) the left (resp. right) multiplication determined by  $a$ , and for any regular element  $u$  of  $A$ , we denote by  $\langle u \rangle$  the inner automorphism  $u_l \cdot u_r^{-1}$  of  $A$ . Moreover, for any subset  $E$  of  $A$  and for any set  $\mathfrak{C}$  of (ring) automorphisms in  $A$ , we shall use the following conventions:  $E_l$  (resp.  $E_r$ ) = the set of left (resp. right) multiplications determined by elements of  $E$ ;  $\langle E \rangle$  = the set of inner automorphisms determined by regular elements of  $E$ ;  $V_A(E)$  = the centralizer of  $E$  in  $A$ ;  $V_A^2(E) = V_A(V_A(E))$ ;  $\mathfrak{F}(E)$  = the group of (ring) automorphisms in  $A$  which leave the elements of  $E$  fixed;  $I(\mathfrak{C})$  = the subring of  $A$  generated by the regular elements  $u$  such that  $\langle u \rangle \in \mathfrak{C}$ ;  $J(\mathfrak{C})$  = the fixring of  $\mathfrak{C}$  in  $A$ . Further, given rings  $S, R$ , we shall understand by an  $S$ - $R$ -module a two-sided module which is treated as a left  $S$ -module and as a right  $R$ -module; and,

if an  $S$ - $R$ -module  $M$  is irreducible as  $S$ - $R$ -module then  $M$  is said to be  $S$ - $R$ -irreducible.

In subsequent use, *subring* of  $A$  will mean subring which contains the identity element 1 of  $A$ . Now, let  $B$  be a subring of  $A$ . If  $A$  is  $B \cdot V_A(B)$ - $A$ -irreducible then  $B$  will be called an  $M$ -subring. Moreover,  $B$  will be called respectively

- simple*, if  $B$  is a simple subring with minimum condition ;
- M-regular* if  $B$  is a simple  $M$ -subring ;
- regular*, if  $B$  and  $V_A(B)$  are both simple.

Next, let  $\mathfrak{H}$  be a group of (ring) automorphisms in  $A$ . If  $\langle I(\mathfrak{H}) \rangle \subset \mathfrak{H}$  then  $\mathfrak{H}$  will be called an  $N$ -group. If  $\mathfrak{H}$  is an  $N$ -group and  $A$  is  $I(\mathfrak{H}) \cdot J(\mathfrak{H})$ - $A$ -irreducible then  $\mathfrak{H}$  will be called an  $M$ -group. Moreover,  $\mathfrak{H}$  will be called respectively

- N-regular*, if  $\mathfrak{H}$  is an  $N$ -group, and  $I(\mathfrak{H})$  is simple ;
- M-regular*, if  $\mathfrak{H}$  is an  $N$ -regular  $M$ -group ;
- regular*, if  $J(\mathfrak{H})$  is regular, and  $\langle V_A(J(\mathfrak{H})) \rangle \subset \mathfrak{H}$ .

Now, we shall make some remarks on the regularity of subrings of  $A$ . First, we shall show

(1) Let  $\mathfrak{B}$  is a subring of  $\mathfrak{A}$  containing  $A_r$ ,  $\mathfrak{B}' = V_{\mathfrak{A}}(\mathfrak{B})$ , and  $B' = \{x \in A; x_i \in \mathfrak{B}'\}$ . Then  $B'$  is a subring of  $A$ , which is anti-isomorphic to  $\mathfrak{B}'$ . If  $A$  is  $\mathfrak{B} \cdot \mathfrak{B}'$ -irreducible then  $B'$  is simple.

In fact, since  $\mathfrak{B} \supset A_r$  our first assertion is obvious. By the minimum condition for right ideals of  $A$ , there exists a minimal  $\mathfrak{B}$ -submodule  $M (\neq 0)$  of  $A$ . If  $A$  is  $\mathfrak{B} \cdot \mathfrak{B}'$ -irreducible then  $A = \sum_{\lambda \in \mathfrak{B}'} M\lambda$ , and  $M\lambda = 0$  or  $M\lambda$  is  $\mathfrak{B}$ -isomorphic to  $M$ . Hence  $A$  is a completely reducible homogeneous  $\mathfrak{B}$ -module, and this is a direct sum of a finite number of irreducible  $\mathfrak{B}$ -submodules each of which is  $\mathfrak{B}$ -isomorphic to  $M$ . Therefore  $\mathfrak{B}'$  is a simple ring with minimum condition.

By (1) we have

(2) If  $B$  is an  $M$ -subring of  $A$  then  $V_A(B)$  is simple, and  $\mathfrak{B}(B)$  is  $M$ -regular. In particular, an  $M$ -regular subring is regular.

(3) If  $\mathfrak{H}$  is an  $M$ -group of automorphisms in  $A$  then  $J(\mathfrak{H})$  is  $M$ -regular.

Finally, we shall give some notions w. r. t. indices for groups and dimensions for modules. For a subset  $E$  of  $A$  and for a subset  $\mathfrak{S}$  of  $\mathfrak{A}$ , we denote by  $E|\mathfrak{S}$  the restriction of  $\mathfrak{S}$  to  $E$ ; and, for a group  $\mathfrak{H}$  of automorphisms in  $A$ , if for  $\sigma, \tau \in \mathfrak{S}$ ,  $E|\sigma = (E|\tau)\delta$  ( $: E \xrightarrow{\delta} A \xrightarrow{\delta} A$ ) for some  $\delta$  in  $\mathfrak{H}$ , then we write  $E|\sigma \sim E|\tau \pmod{\mathfrak{H}}$ . Evidently, the relation  $\sim$  is an equivalence relation in  $E|\mathfrak{S}$ , and the number of the equivalence classess w. r. t.  $\sim$  is denoted as  $(E|\mathfrak{S}:\mathfrak{H})$ . Moreover, for a subgroup  $\mathfrak{H}_1$

of a group  $\mathfrak{S}_2$ , we denote by  $(\mathfrak{S}_2 : \mathfrak{S}_1)$  the index of  $\mathfrak{S}_1$  in  $\mathfrak{S}_2$ . If, for a simple ring  $S$ ,  $M$  is a left (resp. right)  $S$ -module then we denote by  $[M|S]_l$  (resp.  $[M|S]_r$ ) the uniquely determined number of irreducible direct summands of left (resp. right)  $S$ -module  $M$ . In particular, if  $M$  is a left (resp. right)  $S$ -free module then the left (resp. right) dimension is denoted by  $[M:S]_l$  (resp.  $[M:S]_r$ ). If  $B$  is a subring of  $A$  containing a simple subring  $S$  then  $B$  is left (and right)  $S$ -free. Moreover,  $[B:S]_l = [B:S]_r$  if and only if  $[B|S]_l = [B|S]_r$ ; in this case, they are denoted by  $[B:S]$  and by  $[B|S]$  respectively. Finally, for any set  $E$ , we denote by  $\#E$  the cardinal number of  $E$ .

**2. Preliminaries.** Throughout the rest of this paper,  $\mathfrak{S}$ ,  $\mathfrak{S}_1$ ,  $\mathfrak{S}_2$ , etc. will mean groups of (ring) automorphisms in  $A$ . In our study, Jacobson's theory for completely reducible modules [1, Chap. VI] will be used freely. Otherwise, the paper is self-contained.

Now we shall begin our study with the following lemma which contains [7, Lemma 1.3, and Lemma 1.4]; in the same ways as in those proofs, our lemma is proved, however, for the completeness, we shall present the proof.

**Lemma 1.** *Let  $B$  be a subring of  $A$  such that  $A$  is  $B$ - $A$ -irreducible. Then*

(a) *Let  $\sigma$  be a (ring) automorphism in  $A$ . Then, for each non-zero  $a \in A$ ,  $(B|\sigma)a_l A_r$  is  $B_r A_r$ -irreducible and  $x_r \rightarrow (B|\sigma)a_l x_r$  is an  $A_r$ -isomorphism of  $A_r$  onto  $(B|\sigma)a_l A_r$ .*

(b) *Let  $\sigma, \tau$  be (ring) automorphisms in  $A$ , and  $M$  a  $B_r A_r$ -submodule of  $\text{Hom}(B, A)$ . Then  $(B|\sigma)A_r$  is  $B_r A_r$ -isomorphic to  $M$  if and only if  $M = (B|\sigma)a_l A_r$  for some non-zero  $a \in A$ . Moreover,  $(B|\sigma)A_r$  is  $B_r A_r$ -isomorphic to  $(B|\tau)A_r$  if and only if  $B|\sigma \sim B|\tau \pmod{\langle A \rangle}$ .*

(c) *For any subset  $W$  of  $A$ ,  $B|W_l$  is linearly independent over  $A_r$  if and only if  $W$  is linearly right-independent over  $V_A(B)$ . Particularly, in case  $W$  is consisting of regular elements,  $B|\langle W \rangle$  is linearly independent over  $A_r$  if and only if  $W$  is linearly right-independent over  $V_A(B)$ .*

*Proof.* (a): Let  $a, x$  be arbitrary non-zero elements of  $A$ . Then, by our assumption, there holds that  $B_r(B|\sigma)a_l x_r A_r = (B|\sigma)a_l(B\sigma \cdot xA)_r = (B|\sigma)a_l((B \cdot x\sigma^{-1} \cdot A)\sigma)_r = (B|\sigma)a_l A_r$ , whence our assertion is obvious.

(b): If  $(B|\sigma)A_r$  is  $B_r A_r$ -isomorphic to  $M$  and if  $B|\sigma \leftrightarrow \varepsilon$  under the above isomorphism then  $x_r \varepsilon \leftrightarrow x_r(B|\sigma) = (B|\sigma)(x\sigma)_r \leftrightarrow \varepsilon(x\sigma)_r$  for each  $x \in B$ . Hence  $x_r \varepsilon = \varepsilon(x\sigma)_r$ , and so,  $x\varepsilon = 1(x_r \varepsilon) = 1(\varepsilon(x\sigma)_r) = (1\varepsilon)(x\sigma) = x\sigma(1\varepsilon)_l$ . Therefore  $M = \varepsilon A_r = (B|\sigma)a_l A_r$  with  $a = 1\varepsilon (\neq 0)$ . The converse part is an easy

consequence of (a). If  $(B|\sigma)A_r$  is  $B_r$ - $A_r$ -isomorphic to  $(B|\tau)A_r$  then  $(B|\tau)A_r = (B|\sigma)a_iA_r$  for some non-zero element  $a$  of  $A$ . Hence  $B|\tau = (B|\sigma)a_ib_r$  for some  $b \in A$ . This means  $1 = ab$ ; whence  $a$  is regular and  $B|\tau = (B|\sigma)\langle a \rangle$ . The converse is obvious.

(c): Assume that  $B|W_i$  is linearly dependent over  $A_r$  and  $\sum_1^m (B|v_{i1})x_{ir} = 0$  ( $v_i \in W$ ,  $x_i \neq 0 \in A$ ) is a non-trivial relation of the shortest length. Then, by (a), without loss of generality, we may set  $x_1 = -1$ . Hence, we may assume that  $B|v_{i1} = \sum_2^m (B|v_{i2})x_{ir}$ . Now, for arbitrary  $y \in B$  there holds

$$0 = y_r(B|v_{i1}) - (B|v_{i1})y_r = \sum_2^m (B|v_{i2})(yx_i - x_iy)_r.$$

Since our relation is of the shortest length, it follows that  $x_i \in V_A(B)$  ( $i = 2, \dots, m$ ). Consequently, we have  $v_i = \sum_2^m v_ix_i$  ( $x_i \in V_A(B)$ ). The converse will be almost trivial. For the last assertion, note that in case  $W$  is consisting of regular elements,  $B|\langle W \rangle$  is linearly independent over  $A_r$  if and only if so is  $B|W_i$ .

**Corollary 1.** *Let  $B$  be a subring of  $A$ . Then*

(a) *if  $A$  is  $B$ - $A$ -irreducible and if  $B'$  is a subring of  $B$  such that  $B$  is a free left  $B'$ -module with base  $X$  then*

$$\#X \geq (B|\mathfrak{S}(B') : \langle V_A(B') \rangle) [V_A(B') : V_A(B)]_r,$$

(b) *if  $\mathfrak{S}$  is an  $M$ -group such that  $J(\mathfrak{S}) \subset B$  then  $J(\mathfrak{S})$  is regular, and*

$$[B : J(\mathfrak{S})]_i = [(B|\mathfrak{S})A_r | A_r]_r / [A | A],$$

*provided we do not distinguish between two infinite cardinal numbers.*

*Proof.* For (a) note that

$$\#X = [\text{Hom}_{B'}(B, A) | A_r]_r / [A | A] \geq [(B|\mathfrak{S}(B'))A_r : A_r]_r.$$

Then, from Lemma 1, our inequality readily follows. If  $\mathfrak{S}$  is an  $M$ -group then  $J(\mathfrak{S})$  is regular as in the remarks (2)–(3) of § 1. Our second assertion (b) is a direct consequence of [1, Th. VI. 2.1 (Density theorem for completely reducible modules)].

**Corollary 2.** *Let  $\mathfrak{S}$  be an  $M$ -group. If  $\mathfrak{S}'$  is a subgroup of  $\mathfrak{S}$  then*

$$(\mathfrak{S} : \mathfrak{S}') \geq [J(\mathfrak{S}') : J(\mathfrak{S})]_i,$$

*provided we do not distinguish between two infinite cardinal numbers.*

*Proof.* We set  $B = J(\mathfrak{S})$ ,  $B' = J(\mathfrak{S}')$ , and  $\mathfrak{S} = \mathfrak{S}'\sigma_1 \cup \dots \cup \mathfrak{S}'\sigma_n$ , where  $n = (\mathfrak{S} : \mathfrak{S}')$ . Then

$$(B'|\mathfrak{S})A_r = \sum_1^n (B'|\sigma_i)A_r.$$

Hence, by Coro. 1 (b), we obtain  $[B' : B]_i = [(B'|\mathfrak{S})A_r | A_r]_r / [A | A] \leq n$ .

The following lemma contains the result of [6, Lemma 2].

**Lemma 2.** *Let  $S$  be a subset of  $A$ , and  $\mathcal{E} = \mathfrak{F}(S)$ . If  $\mathfrak{N}$  is a normal subgroup of a group  $\mathfrak{H}$  then  $(S|\mathfrak{H}:\mathfrak{N}) = (\mathfrak{H}:(\mathfrak{H}\cap\mathcal{E})\mathfrak{N})$ .*

*Proof.* Let  $\sigma, \tau \in \mathfrak{H}$ . Since  $(\mathfrak{H}\cap\mathcal{E})\mathfrak{N}\sigma = (\mathfrak{H}\cap\mathcal{E})\mathfrak{N}$ , we have  $S|(\mathfrak{H}\cap\mathcal{E})\mathfrak{N}\sigma = S|(\mathfrak{H}\cap\mathcal{E})\mathfrak{N} = (S|\sigma)\mathfrak{N}$ . Hence, if  $(\mathfrak{H}\cap\mathcal{E})\mathfrak{N}\sigma = (\mathfrak{H}\cap\mathcal{E})\mathfrak{N}\tau$  then  $(S|\sigma)\mathfrak{N} = S|(\mathfrak{H}\cap\mathcal{E})\mathfrak{N}\sigma = S|(\mathfrak{H}\cap\mathcal{E})\mathfrak{N}\tau = (S|\tau)\mathfrak{N}$ . This implies  $S|\sigma \sim S|\tau \pmod{\mathfrak{N}}$ . Next, suppose  $S|\sigma \sim S|\tau \pmod{\mathfrak{N}}$ . Then  $S|\sigma = (S|\tau)\delta$  for some  $\delta \in \mathfrak{N}$ , and this is equal to  $S|\tau\delta$ . Hence there exists an element  $\varepsilon \in \mathfrak{F}(S) = \mathcal{E}$  such that  $\sigma = \varepsilon(\tau\delta)$ . Clearly  $\varepsilon \in \mathfrak{H}\cap\mathcal{E}$ ,  $\sigma = \varepsilon(\tau\delta\tau^{-1})\tau$  and  $\varepsilon(\tau\delta\tau^{-1}) \in (\mathfrak{H}\cap\mathcal{E})\mathfrak{N}$ . Thus we obtain  $(\mathfrak{H}\cap\mathcal{E})\mathfrak{N}\sigma = (\mathfrak{H}\cap\mathcal{E})\mathfrak{N}\tau$ . We have therefore proved that  $(\mathfrak{H}:(\mathfrak{H}\cap\mathcal{E})\mathfrak{N}) = (S|\mathfrak{H}:\mathfrak{N})$ .

**3. The notion of reduced indices for  $N$ -groups.** In [6], we introduced the notion of reduced indices for  $N$ -groups of automorphisms in division rings. Now, we shall generalize this notion to simple rings as follows:

**Definition.** For an  $N$ -regular group  $\mathfrak{H}_1$  and for an  $N$ -group  $\mathfrak{H}_2$  containing  $\mathfrak{H}_1$ , we denote by  $[\mathfrak{H}_2||\mathfrak{H}_1]_r$  (resp.  $[\mathfrak{H}_2||\mathfrak{H}_1]_l$ ) the product  $(\mathfrak{H}_2:\mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle)[I(\mathfrak{H}_2):I(\mathfrak{H}_1)]_r$  (resp.  $(\mathfrak{H}_2:\mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle)[I(\mathfrak{H}_2):I(\mathfrak{H}_1)]_l$ ), which is called the *right* (resp. *left*) *reduced index* of  $\mathfrak{H}_1$  in  $\mathfrak{H}_2$ . In case  $[\mathfrak{H}_2||\mathfrak{H}_1]_r = [\mathfrak{H}_2||\mathfrak{H}_1]_l$ , they are denoted by  $[\mathfrak{H}_2||\mathfrak{H}_1]$ .

If  $\mathfrak{H}_1 = \{1\}$  then  $[\mathfrak{H}_2||\{1\}] = (\mathfrak{H}_2:\langle I(\mathfrak{H}_2)\rangle)[I(\mathfrak{H}_2):C]$ , where  $C$  is the center of  $A$ . This is the reduced order of  $\mathfrak{H}_2$  which has been introduced in [1].

For reduced indices, we have the following lemma which contains the result of [6, Lemma 3].

**Lemma 3.** *Let  $\mathfrak{H}_1 \subset \mathfrak{H}_2 \subset \mathfrak{H}_3$  be  $N$ -groups. Then*

(a) *If  $\mathfrak{H}_1, \mathfrak{H}_2$  are  $N$ -regular then*

$$[\mathfrak{H}_3||\mathfrak{H}_1]_r = [\mathfrak{H}_3||\mathfrak{H}_2]_r [\mathfrak{H}_2||\mathfrak{H}_1]_r.$$

(b) *If  $\mathfrak{H}_1$  is  $N$ -regular and  $[\mathfrak{H}_2||\mathfrak{H}_1]_r = [\mathfrak{H}_3||\mathfrak{H}_1]_r < \infty$  then*

$$\mathfrak{H}_2 = \mathfrak{H}_3.$$

*Proof.* (a): Since  $\langle I(\mathfrak{H}_3)\rangle$  is a normal subgroup of  $\mathfrak{H}_3$  and  $\langle I(\mathfrak{H}_3)\rangle \cap \mathfrak{H}_2 = \langle I(\mathfrak{H}_2)\rangle$ , we have

$$\begin{aligned} [\mathfrak{H}_3||\mathfrak{H}_1]_r &= (\mathfrak{H}_3:\mathfrak{H}_1\langle I(\mathfrak{H}_3)\rangle)[I(\mathfrak{H}_3):I(\mathfrak{H}_1)]_r \\ &= (\mathfrak{H}_3:\mathfrak{H}_2\langle I(\mathfrak{H}_3)\rangle)(\mathfrak{H}_2:\mathfrak{H}_1\langle I(\mathfrak{H}_3)\rangle)[I(\mathfrak{H}_3):I(\mathfrak{H}_1)]_r \\ &= (\mathfrak{H}_3:\mathfrak{H}_2\langle I(\mathfrak{H}_3)\rangle)(\mathfrak{H}_2:\mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle) \cdot \\ &\quad [I(\mathfrak{H}_3):I(\mathfrak{H}_2)]_r [I(\mathfrak{H}_2):I(\mathfrak{H}_1)]_r \end{aligned}$$

$$= [\mathfrak{H}_2 || \mathfrak{H}_3]_r [\mathfrak{H}_2 || \mathfrak{H}_1]_r.$$

(b): Note that

$$\begin{aligned} [I(\mathfrak{H}_2) : I(\mathfrak{H}_1)]_r &= [\mathfrak{H}_2 || \mathfrak{H}_1]_r / (\mathfrak{H}_2 : \mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle) \\ &= [\mathfrak{H}_2 || \mathfrak{H}_1]_r / (\mathfrak{H}_2 \langle I(\mathfrak{H}_2) \rangle : \mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle) \\ &= (\mathfrak{H}_2 : \mathfrak{H}_2 \langle I(\mathfrak{H}_2) \rangle) [I(\mathfrak{H}_2) : I(\mathfrak{H}_1)]_r. \end{aligned}$$

Then  $I(\mathfrak{H}_2) = I(\mathfrak{H}_3)$  and  $(\mathfrak{H}_3 : \mathfrak{H}_2 \langle I(\mathfrak{H}_3) \rangle) = 1$ . Hence  $\mathfrak{H}_2 = \mathfrak{H}_3$ .

Moreover, we have the following

**Lemma 4.** Let  $\mathfrak{H}_1 \subset \mathfrak{H}_2$  and  $\mathfrak{H}'_2 \subset \mathfrak{H}_2$  be  $N$ -groups. If  $\mathfrak{H}_1$  is  $N$ -regular and  $I(\mathfrak{H}'_2) = I(\mathfrak{H}_2)$  then  $\mathfrak{H}'_2 \cap \mathfrak{H}_1$  is  $N$ -regular,  $[\mathfrak{H}'_2 || \mathfrak{H}'_2 \cap \mathfrak{H}_1]_r \leq [\mathfrak{H}_2 || \mathfrak{H}_1]_r$ , and if, in addition,  $[\mathfrak{H}'_2 || \mathfrak{H}'_2 \cap \mathfrak{H}_1]_r = [\mathfrak{H}_2 || \mathfrak{H}_1]_r$ , then  $\mathfrak{H}_2 = \mathfrak{H}'_2 \mathfrak{H}_1 = \mathfrak{H}_1 \mathfrak{H}'_2$ .

*Proof.* Since  $I(\mathfrak{H}'_2 \cap \mathfrak{H}_1) = I(\mathfrak{H}_1)$ ,  $\mathfrak{H}'_2 \cap \mathfrak{H}_1$  is  $N$ -regular. Clearly  $(\mathfrak{H}'_2 \cap \mathfrak{H}_1) \langle I(\mathfrak{H}_2) \rangle \subset \mathfrak{H}'_2 \cap \mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle$ . Conversely, if  $\sigma \in \mathfrak{H}'_2 \cap \mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle$  then  $\sigma = \sigma_2 = \sigma_1 \langle v \rangle$ , where  $\sigma_2 \in \mathfrak{H}'_2$ ,  $\sigma_1 \in \mathfrak{H}_1$ , and  $\langle v \rangle \in \langle I(\mathfrak{H}_2) \rangle$ ; and then,  $\sigma_1 = \sigma_2 \langle v \rangle^{-1} \in \mathfrak{H}'_2$ ; whence  $\sigma \in (\mathfrak{H}'_2 \cap \mathfrak{H}_1) \langle I(\mathfrak{H}_2) \rangle$ . Thus, we obtain  $\mathfrak{H}'_2 \cap \mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle = (\mathfrak{H}'_2 \cap \mathfrak{H}_1) \langle I(\mathfrak{H}_2) \rangle$ . From this, the rest of our assertion follows immediately.

**4. On free dimensions and reduced indices.** Firstly, we shall prove the following

**Proposition 1.** Let  $B_1$  be a subring of  $A$ , and  $B_2$  an  $M$ -subring of  $A$  containing  $B_1$  which is a free left  $B_1$ -module with base  $X$ . If  $\mathfrak{H}$  is a subgroup of  $\mathfrak{F}(B_1)$  containing  $\langle V_A(B_1) \rangle$  then

$$\#X \geq [\mathfrak{H} || \mathfrak{H} \cap \mathfrak{F}(B_2)]_r$$

provided we do not distinguish between two infinite cardinal numbers.

*Proof.* Let  $\{f_{ij}; 1 \leq i, j \leq n\}$  be a system of matrix units of  $V_A(B_2)$  such that the centralizer of  $\{f_{ij}\}$  in  $V_A(B_2)$  is a division ring. Then  $B_3 = \sum_{i,j} B_2 f_{ij}$  is a free left  $B_1$ -module with base  $\{x_k f_{ij}; x_k \in X, 1 \leq i, j \leq n\}$ , and  $A$  is  $B_3$ - $A$ -irreducible. Then

$$\begin{aligned} (1) \quad n^2 \cdot \#X &\geq (B_3 | \mathfrak{H} : \langle V_A(B_1) \rangle) [V_A(B_1) : V_A(B_3)]_r && (\text{Coro. 1(a)}) \\ &= (\mathfrak{H} : (\mathfrak{H} \cap \mathfrak{F}(B_3)) \langle V_A(B_1) \rangle) [V_A(B_1) : V_A(B_3)]_r && (\text{Lemma 2}) \\ &= [\mathfrak{H} || \mathfrak{H} \cap \mathfrak{F}(B_3)]_r. \end{aligned}$$

If, in (1), we take  $B_2$  and  $\mathfrak{H} \cap \mathfrak{F}(B_2)$  instead of  $B_1$  and  $\mathfrak{H}$  respectively, then

$$\begin{aligned} n^2 &\geq [\mathfrak{H} \cap \mathfrak{F}(B_2) || \mathfrak{H} \cap \mathfrak{F}(B_2)]_r \\ &\geq [I(\mathfrak{H} \cap \mathfrak{F}(B_2)) : I(\mathfrak{H} \cap \mathfrak{F}(B_2))]_r = n^2, \end{aligned}$$

and so

$$(2) \quad n^2 = [\mathfrak{H} \cap \mathfrak{F}(B_2) || \mathfrak{H} \cap \mathfrak{F}(B_3)]_r.$$

Hence, from (1) and (2), we obtain

$$(3) \quad \#X \geq [\mathfrak{H} \parallel \mathfrak{H} \cap \mathfrak{I}(B_2)]_r \quad (\text{Lemma 3 (a)}).$$

**Proposition 2.** Let  $B_1$  be a simple subring of  $A$ , and  $B_2$  an  $M$ -subring of  $A$  containing  $B_1$ . If  $\mathfrak{H}$  is a subgroup of  $\mathfrak{I}(B_1)$  containing  $\langle V_A(B_1) \rangle$  such that  $\mathfrak{H}A_r$  is dense in  $V_n(B_n)$  in finite topology then

$$[B_2 : B_1]_t = [\mathfrak{H} \parallel \mathfrak{H} \cap \mathfrak{I}(B_2)]_r,$$

provided we do not distinguish between two infinite cardinal numbers.

*Proof.* Let  $B_3$  be as in the proof of Prop. 1. Then

$$\begin{aligned} n^2[B_2 : B_1]_t &= [(B_3 | \mathfrak{H})A_r : A_r]_r \\ &= (B_3 | \mathfrak{H} : \langle V_A(B_1) \rangle) [V_A(B_1) : V_A(B_2)]_r \end{aligned} \quad (\text{Lemma 1}).$$

Hence, in our case, the equalities of (1) (as in the proof of Prop. 1) hold for all, and so, the equality of (3) holds. This means our assertion.

**Theorem 1.** Let  $B_1$  be a simple subring of  $A$ , and  $B_2$  an  $M$ -subring of  $A$  containing  $B_1$ . Let  $\mathfrak{H}_1$  be an  $N$ -regular group, and  $\mathfrak{H}_2$  an  $M$ -group containing  $\mathfrak{H}_1$ . Then

$$\begin{aligned} (a) \quad & [B_2 : B_1]_t \geq [\mathfrak{I}(B_1) \parallel \mathfrak{I}(B_2)]_r, \\ (b) \quad & [\mathfrak{H}_2 \parallel \mathfrak{H}_1]_r \geq [J(\mathfrak{H}_1) : J(\mathfrak{H}_2)]_t, \end{aligned}$$

provided we do not distinguish between two infinite cardinal numbers.

*Proof.* Our assertion (a) is a direct consequence of Prop. 1. We set  $S_i = J(\mathfrak{H}_i)$  ( $i=1, 2$ ), and  $S = J(\mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle)$ . Then

$$[S : S_2]_t \leq (\mathfrak{H}_2 : \mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle) \quad (\text{Coro. 2}).$$

Since  $\mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle$  is an  $M$ -group, we have

$$\begin{aligned} [S_1 : S]_t &= [(S_1 | \mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle)A_r : A_r]_r / [A : A] \\ &= [(S_1 | \langle I(\mathfrak{H}_2) \rangle)A_r : A_r]_r / [A : A] \\ &= [(S_1 | I(\mathfrak{H}_2)_t)A_r : A_r]_r / [A : A] \\ &\leq [I(\mathfrak{H}_2) : I(\mathfrak{H}_1)]_r \end{aligned} \quad (\text{Note that } I(\mathfrak{H}_1) \subset V_A(S_1)).$$

Therefore

$$\begin{aligned} [S_1 : S_2]_t &= [S_1 : S]_t [S : S_2]_t \\ &\leq [I(\mathfrak{H}_2) : I(\mathfrak{H}_1)]_r (\mathfrak{H}_2 : \mathfrak{H}_1 \langle I(\mathfrak{H}_2) \rangle) = [\mathfrak{H}_2 \parallel \mathfrak{H}_1]_r. \end{aligned}$$

**Theorem 2.** Let  $B_1$  be an  $M$ -regular subring of  $A$ , and  $B_2$  an  $M$ -subring of  $A$  containing  $B_1$ . Let  $\mathfrak{H}_1$  be an  $M$ -regular group, and  $\mathfrak{H}_2$  an  $M$ -group containing  $\mathfrak{H}_1$ . Then

(a)  $[B_2 : B_1]_t \geq [\mathfrak{I}(B_1) \parallel \mathfrak{I}(B_2)]_r = [J(\mathfrak{I}(B_2)) : J(\mathfrak{I}(B_1))]_t$  (If  $J(\mathfrak{I}(B_1)) = B_1$  then the equalities hold for all.),

(b)  $[\mathfrak{H}_2 \parallel \mathfrak{H}_1]_r \geq [J(\mathfrak{H}_1) : J(\mathfrak{H}_2)]_t = [\mathfrak{I}(J(\mathfrak{H}_2)) \parallel \mathfrak{I}(J(\mathfrak{H}_1))]_r$  (If  $\mathfrak{I}(J(\mathfrak{H}_1)) =$

$\mathfrak{S}_1$  then the equalities hold for all.),  
provided we do not distinguish between two infinite cardinal numbers.

*Proof.* Clearly  $\mathfrak{S}(B_i)$ ,  $J(\mathfrak{S}(B_i))$ ,  $J(\mathfrak{S}_i)$ , and  $\mathfrak{S}(J(\mathfrak{S}_i))$  ( $i=1, 2$ ) are all  $M$ -regular, and moreover,  $\mathfrak{S}(B_i) = \mathfrak{S}(J(\mathfrak{S}(B_i)))$ ,  $J(\mathfrak{S}_i) = J(\mathfrak{S}(J(\mathfrak{S}_i)))$  ( $i=1, 2$ ). Hence the theorem follows from Th. 1.

**5. The regularity and duality of  $M$ -subrings and  $M$ -groups.** The following proposition contains the result of [8, Th. 1].

**Proposition 3.** Let  $B_1$  be a simple subring of  $A$ , and  $B_2$  an  $M$ -subring of  $A$  containing  $B_1$  which is left finite over  $B_1$ . Suppose that  $\mathfrak{S}(B_1)$  contains a subgroup  $\mathfrak{S}$  containing  $\langle V_A(B_1) \rangle$  such that  $\mathfrak{S}A_r$  is dense in  $V_{\mathfrak{N}}(B_{11})$  in finite topology. Then the following hold :

(a)  $B_2 = J(\mathfrak{S} \cap \mathfrak{S}(B_2))$ , and  $B_2$  is regular.

Let  $B$  be an intermediate ring of  $B_2/B_1$ .

(b) If  $B$  is simple then  $A$  is a completely reducible  $B$ - $A$ -module.

(c)  $B$  is an  $M$ -subring of  $A$  if and only if  $B$  is regular.

*Proof.* (a): We set  $B'_2 = J(\mathfrak{S} \cap \mathfrak{S}(B_2))$ . Then  $B_2 \subset B'_2$ ,  $\mathfrak{S} \cap \mathfrak{S}(B_2) = \mathfrak{S} \cap \mathfrak{S}(B'_2)$ , and  $B'_2$  is an  $M$ -subring of  $A$ . Hence, by Poop. 2, we have

$$[B_2 : B_1]_l = [\mathfrak{S} \cap \mathfrak{S}(B_2) : \mathfrak{S}(B_2)]_r = [\mathfrak{S} \cap \mathfrak{S}(B'_2) : \mathfrak{S}(B'_2)]_r = [B'_2 : B_1]_l.$$

Therefore  $B_2 = B'_2$ . Then  $B_2$  is regular as in the remarks (2)–(3) of § 1.

(b): If  $\{f_{ij}\}$  is a system of matrix units of  $V_A(B_2)$  such that the centralizer of  $\{f_{ij}\}$  in  $V_A(B_2)$  is a division ring then  $B_3 = \sum_{i,j} B_2 f_{ij}$  is an  $M$ -subring of  $A$  containing  $B_2$  such that  $[B_3 : B_1]_l < \infty$  and  $A$  is  $B_3$ - $A$ -irreducible. Hence, in case that  $A$  is  $B_2$ - $A$ -irreducible, it suffices to prove our assertion; whence, we may assume that  $A$  is  $B_2$ - $A$ -irreducible. Let  $B$  be a simple intermediate ring of  $B_2/B_1$ , and let  $BaA$  be a minimal  $B$ - $A$ -submodule of  $A$  such that the composition series of  $BaA$  as  $A$ -module is of the shortest length among minimal  $B$ - $A$ -submodules of  $A$ . Since  $B_2$  is a simple ring by (a), there exists an element  $b$  of  $A$  such that  $B_2 b \ni a$ , and  $B_2 b$  is  $B_2$ -isomorphic to  $B_2$  under the mapping  $\varepsilon : xb \rightarrow x (x \in B_2)$ . Then  $\text{Hom}_{B_1}(B_2 b, A) = \varepsilon \varepsilon^{-1} \text{Hom}_{B_1}(B_2 b, A) \subset \varepsilon \text{Hom}_{B_1}(B_2, A) \subset \text{Hom}_{B_1}(B_2 b, A)$ ; and this gives  $\varepsilon \text{Hom}_{B_1}(B_2, A) = \text{Hom}_{B_1}(B_2 b, A)$ . Hence  $Ba | \varepsilon \text{Hom}_{B_1}(B_2, A) = Ba | \text{Hom}_{B_1}(B_2 b, A) = \text{Hom}_{B_1}(Ba, A)$ , and there follows

$$Ba(\varepsilon \text{Hom}_{B_1}(B_2, A)) = A.$$

Since  $\mathfrak{S}(B_2)A_r$  is dense in  $V_{\mathfrak{N}}(B_{21})$  by (a), there exists an element  $\varepsilon^* \in \mathfrak{S}(B_2)A_r$  such that  $B_2 b | \varepsilon^* = \varepsilon$ ;  $\varepsilon^* = \sum_{i=1}^m \sigma_i a_i$ ,  $\sigma_i \in \mathfrak{S}(B_2)$ ,  $a_i \in A (i=1, \dots, m)$ . Moreover, since  $\text{Hom}_{B_1}(B_2, A) \subset \text{Hom}_{B_1}(B_2, A) = B_2 | \mathfrak{S}(B_1)A_r$ , by Lemma 1,

we may write  $\text{Hom}_{B_1}(B_2, A) = B_2 | \sum_1^n \tau_j u_{j1} A_r$ ,  $\tau_j \in \mathfrak{S}(B_1)$ ,  $u_j \in A$  ( $j=1, \dots, n$ ). Therefore it follows that

$$\begin{aligned} A &= Ba(\sum_i \sigma_i a_r)(\sum_j \tau_j u_{j1} A_r) \\ &= \sum_{i,j} (Ba) \sigma_i \tau_j u_{j1} A_r = \sum_{i,j} u_j ((Ba) \sigma_i \tau_j) A. \end{aligned}$$

Since  $u_j((Ba) \sigma_i \tau_j) A = u_j((BaA) \sigma_i \tau_j)$ , we have

$$[u_j((Ba) \sigma_i \tau_j) A | A]_r \leq [BaA | A]_r.$$

Furthermore, as  $B_2 | \tau_j u_{j1} \in \text{Hom}_{B_1}(B_2, A)$ , we have that for every  $x \in B$ ,

$$u_j(x \tau_j) = x(\tau_j u_{j1}) = (x1)(\tau_j u_{j1}) = x(1 \tau_j u_{j1}) = x u_{j1}, \text{ and hence,}$$

$$u_j((Ba) \sigma_i \tau_j) A = u_j(B \tau_j)(a \sigma_i \tau_j) A = B u_j(a \sigma_i \tau_j) A.$$

Therefore  $u_j((Ba) \sigma_i \tau_j) A$  is 0 or an irreducible  $B$ - $A$ -module. Thus  $A$  is completely reducible as  $B$ - $A$ -module.

(c): Let  $B$  be an intermediate ring of  $B_2/B_1$ . If  $B$  is regular then  $B$  is an  $M$ -subring of  $A$  by (b). Conversely, if  $B$  is an  $M$ -subring of  $A$  then  $B$  is regular by (a).

**Remark 1.** Let  $B$  be a subring of  $A$ , and let  $A$  be completely reducible as  $B$ - $A$ -module. Then  $A$  is completely reducible as  $B \cdot V_A(B)$ - $A$ -module, and the homogeneous components of  $B \cdot V_A(B)$ - $A$ -module  $A$  are irreducible and non-isomorphic. Hence  $A$  is completely reducible as  $V_A(B)$ - $A$ -module. Hence, if  $B = V_A^2(B)$  and, is a simple ring then  $A$  is an irreducible  $B \cdot V_A(B)$ - $A$ -module, and so,  $V_A(B)$  is a simple ring too (, that is,  $B$  is regular) ([1, Th. VI. 1. 1—2]).

**Theorem 3.** Let  $B_1 \subset B_2$  be  $M$ -subrings of  $A$  such that  $J(\mathfrak{S}(B_1)) = B_1$  (then  $B_1$  is regular) and  $[B_2 : B_1]_i < \infty$ . Then an intermediate ring  $B$  of  $B_2/B_1$  is an  $M$ -subring of  $A$  if and only if  $B$  is regular. If the condition holds then  $J(\mathfrak{S}(B)) = B$  and  $[B : B_1]_i = [\mathfrak{S}(B_1) | \mathfrak{S}(B)]_r$ . Particularly so is for  $B_2$ .

*Proof.* As in the remarks (2)—(3) of §1,  $B_1$  is regular and  $M$ -regular. Since  $\mathfrak{S}(B_1) A_r$  is dense in  $V_{B_1}(B_{11})$  in finite topology, the theorem follows from Prop. 2 and Prop. 3. (Our last assertion is too a direct consequence of Th. 2 (a):  $[B_2 : B_1]_i = [\mathfrak{S}(B_1) | \mathfrak{S}(B_2)]_r = [J(\mathfrak{S}(B_2)) : J(\mathfrak{S}(B_1))]_i = [J(\mathfrak{S}(B_2)) : B_1]_i$ .)

The following theorem is similar to the result of [9, Th. 3].

**Theorem 4.** Let  $B_1 \subset B_2$  be  $M$ -subrings of  $A$  such that  $V_A^2(B_1) = B_1$  (then  $B_1$  is regular) and  $[B_2 : B_1]_i < \infty$ . Then an intermediate ring  $B$  of  $B_2/B_1$  is an  $M$ -subring of  $A$  if and only if  $B$  is simple. If the condition holds then  $B$  is regular,  $V_A^2(B) = B$ , and  $[B : B_1]_i = [V_A(B_1) : V_A(B)]_r$ .

Particularly so is for  $B_2$ .

*Proof.* By our assumption,  $\langle V_A(B_1) \rangle A_r (= V_A(B_1)_l A_r)$  is dense in  $V_{\mathfrak{A}}(B_1)$  in finite topology. Hence, by Prop. 2, Prop. 3, and Remark 1, it suffices to prove that  $V_A^2(B) = B$  for every simple intermediate ring  $B$  of  $B_2/B_1$ . Since  $V_A^2(B_2) = J(\langle V_A(B_2) \rangle) = B_2$  (Prop. 3),  $B \subset V_A^2(B) \subset B_2$ . Let  $a$  be an element of  $B_2$  which is not contained in  $B$ . Then, there exists some  $f \in V_{\mathfrak{A}}(B_1)$  such that  $a_i f \neq f a_i$ . Now, let  $\{f_{ij}\}$  be a system of matrix units of  $V_A(B_2)$  such that the centralizer of  $\{f_{ij}\}$  in  $V_A(B_2)$  is a division ring. Set here  $B_3 = \sum_{i,j} B_2 f_{ij}$ . Then  $A$  is  $B_3 A$ -irreducible. Hence, by Lemma 1, we have  $B_3 | V_{\mathfrak{A}}(B_1)_r = B_3 | V_A(B_1)_l A_r \supset B_3 | V_{\mathfrak{A}}(B_1) = (B_3 | u_1)_r A_r \oplus \cdots \oplus (B_3 | u_n)_r A_r$  (direct sum)  $\ni B_3 | f$ . Clearly  $u_i \in V_A(B)$  ( $i=1, \dots, n$ ). Since  $a_i f \neq f a_i$ , there exists some  $u_j$  such that  $au_j \neq u_j a$ . This implies  $V_A^2(B) = B$ .

**Theorem 5.** Let  $\mathfrak{H}_1 \subset \mathfrak{H}_2$  be  $M$ -groups such that  $\mathfrak{Z}(J(\mathfrak{H}_1)) = \mathfrak{H}_1$  (then  $\mathfrak{H}_1$  is regular) and  $[\mathfrak{H}_1 | \mathfrak{H}_1]_r < \infty$ . Then an intermediate group  $\mathfrak{H}$  of  $\mathfrak{H}_2/\mathfrak{H}_1$  is an  $M$ -group if and only if  $\mathfrak{H}$  is regular. If the condition holds then  $\mathfrak{Z}(J(\mathfrak{H})) = \mathfrak{H}$  and  $[\mathfrak{H} | \mathfrak{H}_1]_r = [J(\mathfrak{H}_1) : J(\mathfrak{H})]_l$ . Particularly so is for  $\mathfrak{H}_2$ .

*Proof.* As in the remarks (2)–(3) of § 1,  $\mathfrak{H}_1$  is regular and  $M$ -regular. Then, by Th. 2 (b), we have

$$\begin{aligned} \infty &> [\mathfrak{H}_2 | \mathfrak{H}_1]_r = [J(\mathfrak{H}_1) : J(\mathfrak{H}_2)]_l \\ &= [\mathfrak{Z}(J(\mathfrak{H}_2)) | \mathfrak{Z}(J(\mathfrak{H}_1))]_r = [\mathfrak{Z}(J(\mathfrak{H}_2)) | \mathfrak{H}_1]_r. \end{aligned}$$

Since  $\mathfrak{Z}(J(\mathfrak{H}_2)) \supset \mathfrak{H}_2$ , it follows from Lemma 3 (b) that  $\mathfrak{Z}(J(\mathfrak{H}_2)) = \mathfrak{H}_2$ . Hence  $\mathfrak{H}_2$  is regular. Let  $\mathfrak{H}$  be an intermediate group of  $\mathfrak{H}_2/\mathfrak{H}_1$ . If  $\mathfrak{H}$  is an  $M$ -group then  $[\mathfrak{H} | \mathfrak{H}_1]_r \leq [\mathfrak{H}_2 | \mathfrak{H}_1]_r < \infty$ , and whence  $\mathfrak{H}$  is regular. Conversely, suppose that  $\mathfrak{H}$  is regular. Then  $J(\mathfrak{H})$  is a regular intermediate ring of  $J(\mathfrak{H}_1)/J(\mathfrak{H}_2)$ . Since  $J(\mathfrak{H}_i)$ 's are  $M$ -subrings of  $A$ ,  $J(\mathfrak{H})$  is  $M$ -regular by Th. 3. Hence  $\mathfrak{H}$  is an  $M$ -group. Then  $\mathfrak{Z}(J(\mathfrak{H})) = \mathfrak{H}$  and  $[\mathfrak{H} | \mathfrak{H}_1]_r = [J(\mathfrak{H}_1) : J(\mathfrak{H})]_l$ .

**Corollary 3.** Let  $\mathfrak{H}_1 \subset \mathfrak{H}_2$  be  $M$ -groups such that  $\mathfrak{Z}(J(\mathfrak{H}_1)) = \mathfrak{H}_1$  and  $[\mathfrak{H}_2 | \mathfrak{H}_1]_r < \infty$ , and let  $\mathfrak{H}'_2$  be a subgroup of  $\mathfrak{H}_2$  such that  $\mathfrak{H}'_2 \supset \langle I(\mathfrak{H}_2) \rangle$  and  $J(\mathfrak{H}'_2) = J(\mathfrak{H}_2)$ . If  $\mathfrak{H}$  is a regular intermediate group of  $\mathfrak{H}_2/\mathfrak{H}_1$  then  $J(\mathfrak{H}) = J(\mathfrak{H}'_2 \cap \mathfrak{H})$ ,  $[J(\mathfrak{H}) : J(\mathfrak{H}_2)]_l = [\mathfrak{H}_2 | \mathfrak{H}]_r = [\mathfrak{H}'_2 | \mathfrak{H}_2 \cap \mathfrak{H}]_r = [J(\mathfrak{H}'_2 \cap \mathfrak{H}) : J(\mathfrak{H}_2)]_l$ ,  $[\mathfrak{H} | \mathfrak{H}_1]_r = [\mathfrak{H}'_2 \cap \mathfrak{H} | \mathfrak{H}_2 \cap \mathfrak{H}_1]_r$ , and  $\mathfrak{H}_2 = \mathfrak{H}'_2 \mathfrak{H} = \mathfrak{H} \mathfrak{H}'_2$ .

*Proof.* By Th. 5, we have  $\mathfrak{Z}(J(\mathfrak{H})) = \mathfrak{H}$ . Since  $\mathfrak{H}'_2 A_r$  is dense in  $V_{\mathfrak{A}}(J(\mathfrak{H}'_2)_l)$  in finite topology and  $[J(\mathfrak{H}) : J(\mathfrak{H}_2)]_l = [J(\mathfrak{H}) : J(\mathfrak{H}_2)]_l = [\mathfrak{H}_2 | \mathfrak{H}]_r < \infty$  (Th. 5), it follows from Prop. 3 that  $J(\mathfrak{H}) = J(\mathfrak{H}'_2 \cap \mathfrak{Z}(J(\mathfrak{H})) = J(\mathfrak{H}'_2 \cap \mathfrak{H})$ . Combining this with Prop. 2 and Th. 5, we obtain our second and third assertions. Our last assertion is a direct consequence of Lemma 4.

**Remark 2.** Let  $B$  and  $I$  be  $M$ -subrings of  $A$  such that  $B_0 = V_A(I) \subset B$  and  $I_0 = V_A(B) \subset I$ . Then  $B_0$  and  $I_0$  are regular, and we have the following (a)—(c').

$$(a) \quad [B : B_0]_t = [I : I_0]_r,$$

$$(a') \quad [B : B_0]_r = [I : I_0]_t,$$

provided we do not distinguish between two infinite dimensions.

(b) If one of (a) and (a') is of finite dimension then  $V_A^2(B) = B$ ,  $V_A^2(I) = I$  and, are regular.

(c) If (a) is of finite dimension then; an intermediate ring  $B'$  of  $B/B_0$  is an  $M$ -subring of  $A$  if and only if  $B'$  is simple (Th. 4); whence, an intermediate ring  $I'$  of  $I/I_0$  is an  $M$ -subring of  $A$  if and only if  $V_A(I')$  is simple and  $V_A^2(I') = I'$ .

(c') If (a') is of finite dimension then there holds the statement which is symmetric to (c).

In fact, as in the remark (2) of § 1,  $B_0$  and  $I_0$  are regular. Then, by Th. 1, we have

$$[B : B_0]_t \geq [V_A(B_0) : I_0]_r \geq [I : I_0]_r.$$

Moreover, since  $I_t A_r$  is a subring of  $\mathfrak{A}$  which is dense in  $V_{\mathfrak{A}}(B_0)_t$  in finite topology, we have

$$[I : I_0]_r \geq [(B | I_t) A_r | A_r]_r / [A | A] = [B : B_0]_t.$$

Combining this with Th. 4, we obtain our assertion.

**6. Relative Galois correspondences.** Our first result is the next theorem which is a combination of Th. 3 and Th. 5.

**Theorem 6.** Let  $\mathfrak{G}_1 \subset \mathfrak{G}_2$  be  $M$ -groups such that  $\mathfrak{J}(J(\mathfrak{G}_1)) = \mathfrak{G}_1$  and  $[\mathfrak{G}_2 | \mathfrak{G}_1]_r < \infty$ . Then there exists a 1—1 dual correspondence between regular intermediate groups of  $\mathfrak{G}_2/\mathfrak{G}_1$  and regular intermediate rings of  $J(\mathfrak{G}_1)/J(\mathfrak{G}_2)$ , in the sense of the following

$$\mathfrak{G} = \mathfrak{J}(B) \leftrightarrow B = J(\mathfrak{G}).$$

**Remark 3.** In view of Th. 3, we have the following

(i) Let  $B_1 \subset B_2$  be  $M$ -subrings of  $A$  such that  $J(\mathfrak{J}(B_1)) = B_1$  and  $[B_2 : B_1]_t < \infty$ . Then there exists a 1—1 dual correspondence between regular intermediate rings of  $B_2/B_1$  and regular intermediate groups of  $\mathfrak{J}(B_1)/\mathfrak{J}(B_2)$ , in the sense of that  $B = J(\mathfrak{G}) \leftrightarrow \mathfrak{G} = \mathfrak{J}(B)$ .

Particularly we have the following

(ii) Let  $B$  be an  $M$ -subring of  $A$  such that  $J(\mathfrak{J}(B)) = B$ , and let  $B'$  be a subring of  $A$  containing  $B$  such that  $V_A(B) = V_A(B')$  and  $[B' : B]_t < \infty$ .

$\infty$ . Then  $B'$  is an  $M$ -subring, and there exists a 1—1 dual correspondence between intermediate rings of  $B'/B$  and intermediate groups of  $\mathfrak{F}(B)/\mathfrak{F}(B')$ , in the sense of that  $B''=J(\mathfrak{H}'') \leftrightarrow \mathfrak{H}''=\mathfrak{F}(B'')$ .

**Lemma 5.** *Let  $S \subset R$  be subrings of  $A$  such that  $J(\mathfrak{F}(R))=R$  and, is an  $M$ -subring of  $A$  which is an irreducible  $S$ - $R$ -module. Then  $A$  is a completely reducible  $S$ - $A$ -module. If  $J(\mathfrak{F}(S))=S$  then  $S$  is a simple ring, and  $[R:S]_i = [\mathfrak{F}(S) \parallel \mathfrak{F}(R)]_r$ , provided we do not distinguish between two infinite cardinal numbers.*

*Proof.* Clearly  $A$  is an irreducible  $R$ - $\mathfrak{F}(R)A_r$ -module. Let  $M$  be a (non-zero) minimal  $\mathfrak{F}(R)A_r$ -submodule of  $A$ . Then  $A = \sum_{x \in R} xM$ , and  $xM = 0$  or isomorphic to  $M$  as  $\mathfrak{F}(R)A_r$ -module. Hence  $A$  is a homogeneous completely reducible  $\mathfrak{F}(R)A_r$ -module. Let  $M'$  be a non-zero  $\mathfrak{F}(R)A_r$ -submodule of  $A$ . Then  $A = M' \oplus M''$  (direct sum) for some  $\mathfrak{F}(R)A_r$ -submodule  $M''$  of  $A$ , and then  $1 = e' + e''$ , where  $0 \neq e' \in M'$ ,  $e'' \in M''$ . For every  $\sigma \in \mathfrak{F}(R)$ , we have  $1 = 1\sigma = e'\sigma + e''\sigma$ ,  $e'\sigma \in M'$ ,  $e''\sigma \in M''$ , and whence,  $e'\sigma = e'$ . Since  $J(\mathfrak{F}(R))=R$ , it follows that  $e' \in M' \cap R$ . Thus  $M' \cap R \neq \{0\}$ . If  $N$  is a non-zero  $S$ - $\mathfrak{F}(R)A_r$ -submodule of  $A$  then  $N$  is an  $\mathfrak{F}(R)A_r$ -module, and whence, there exists some non-zero element  $a$  of  $N \cap R$ . From this, we have  $1 \in R = SaR \subset N$ . Thus  $N=A$ . Therefore  $A$  is an irreducible  $S$ - $\mathfrak{F}(R)A_r$ -module. Let  $SaA$  a minimal  $S$ - $A$ -submodule of  $A$  such that the composition series of  $BaA$  is of the shortest length among minimal  $S$ - $A$ -submodules of  $A$ . Then  $A = \sum_{\sigma \in \mathfrak{F}(R)} S(a\sigma)A$ , and  $[S(a\sigma)A | A]_r = [SaA | A]_r$ . Hence  $A$  is a completely reducible  $S$ - $A$ -module. Clearly  $A$  is an irreducible  $S$ - $\mathfrak{F}(S)A_r$ -module and, hence, is a homogeneous completely reducible  $\mathfrak{F}(S)A_r$ -module. Hence, if  $J(\mathfrak{F}(S))=S$  then  $S = V_{\mathfrak{A}}(\mathfrak{F}(S)A_r)$ , which is a simple ring, and  $\mathfrak{F}(S)A_r$  is dense in  $V_{\mathfrak{A}}(S_i)$  in finite topology; then, by Prop. 2, we have  $[R:S]_i = [\mathfrak{F}(S) \parallel \mathfrak{F}(R)]_r$ .

The following theorem contains the fundamental theorem in finite outer Galois theory.

**Theorem 7.** *Let  $\mathfrak{H}$  be an  $M$ -group such that  $\mathfrak{F}(J(\mathfrak{H}))=\mathfrak{H}$ , and let  $\mathfrak{H}'$  be a group containing  $\mathfrak{H}$  such that  $I(\mathfrak{H}')=I(\mathfrak{H})$  and  $(\mathfrak{H}':\mathfrak{H})<\infty$ . Then  $\mathfrak{H}'$  is an  $M$ -group, and there exists a 1—1 dual correspondence between intermediate groups of  $\mathfrak{H}'/\mathfrak{H}$  and intermediate rings of  $J(\mathfrak{H})/J(\mathfrak{H}')$ , in the sense of the following*

$$\mathfrak{H}''=\mathfrak{F}(B'') \quad \leftrightarrow \quad B''=J(\mathfrak{H}'').$$

*Proof.* By Th. 6, it suffices to prove that  $\mathfrak{H}'$  is an  $M$ -group. By general theory of groups, there exists a normal subgroup  $\mathfrak{N}$  of  $\mathfrak{H}'$  such that  $\mathfrak{N} \subset \mathfrak{H}$  and  $(\mathfrak{H}':\mathfrak{N})<\infty$ . We set  $S=J(\mathfrak{H}')$ ,  $R=J(\mathfrak{N} \langle I(\mathfrak{H}) \rangle)$ , and  $\mathfrak{N}_0=$

$\mathfrak{F}(J(\mathfrak{N}\langle I(\mathfrak{H}) \rangle))$ . Then  $R$  is simple,  $\mathfrak{N}\langle I(\mathfrak{H}) \rangle \subset \mathfrak{N}_0 \subset \mathfrak{H}$ ,  $(\mathfrak{H}' : \mathfrak{N}_0) < \infty$ ,  $J(\mathfrak{N}_0) = R$ ,  $\mathfrak{F}(J(\mathfrak{N}_0)) = \mathfrak{N}_0$ , and  $\mathfrak{N}_0$  is an  $M$ -group. Since  $\mathfrak{N}\langle I(\mathfrak{H}) \rangle$  is a normal subgroup of  $\mathfrak{H}'$ ,  $R\sigma = R$  for every  $\sigma \in \mathfrak{H}'$ . Hence  $\mathfrak{N}_0$  is a normal subgroup of  $\mathfrak{H}'$  and, since  $\mathfrak{F}(J(\mathfrak{N}_0)) = \mathfrak{N}_0$ ,  $R|\mathfrak{H}'$  is a finite group of automorphisms in  $R$  which is isomorphic to the factor group of  $\mathfrak{H}'$  relative to  $\mathfrak{N}_0$ . If, for  $\sigma \in \mathfrak{H}'$ ,  $R|\sigma = R|\langle a \rangle$  for some regular element  $a$  of  $A$ , that  $\langle a \rangle \sigma^{-1} \in \mathfrak{N}_0 \subset \mathfrak{H}'$ , and so,  $\langle a \rangle \in \mathfrak{H}'\sigma = \mathfrak{H}'$ . This implies  $\langle a \rangle \in I(\mathfrak{H}') = I(\mathfrak{N}_0)$ . Hence  $R|\langle a \rangle = 1$ . Therefore  $R|\mathfrak{H}'$  is a group of outer automorphisms in  $R$ . Then, by finite outer Galois theory, we have

(1)  $S$  is a simple ring,  $[R : S] = (\mathfrak{H}' : \mathfrak{N}_0)$ , and

(2)  $R$  is an irreducible  $S$ - $R$ -module.

However, for the reader's convenience we include the proofs of (1)–(2). We set  $\mathfrak{F} = R|\mathfrak{H}'$ , and let  $\mathfrak{B}$  be a non-zero two-sided ideal of  $\mathfrak{F}R_r$ . Then  $\mathfrak{B}$  contains an irreducible  $R_r$ - $R_r$ -submodule  $M$ . By Lemma 1,  $M = \sigma u_i R_r$  for some  $\sigma \in \mathfrak{F}$  and for some non-zero  $u \in R$ . Then  $u_i R_r \subset \mathfrak{B} \subset \mathfrak{F}R_r$  and  $u_i R_r$  is  $R_r$ - $R_r$ -isomorphic to  $R_r$ . Since  $\mathfrak{F}$  is outer,  $R_r$  is a homogeneous component of  $\mathfrak{F}R_r$ . Hence  $u_i R_r = R_r \ni 1$ , and this implies  $\mathfrak{B} = \mathfrak{F}R_r$ . Therefore  $\mathfrak{F}R_r$  is a simple ring with minimum condition. Then, by general theory of simple rings,  $S$  is simple,  $[R : S]_i = [\mathfrak{F}R_r : R_r]_r$ ,  $\text{Hom}_{S_i}(R, R) = \mathfrak{F}R_r$ , and  $R$  is  $S$ - $\mathfrak{F}R_r$ -irreducible. This contains  $[\mathfrak{F}R_r : R_r]_r = [V_R(S)_i \mathfrak{F}R_r : R_r]_r = [\text{Hom}_{S_i}(R, R) : R_r]_r$ . Since  $\mathfrak{F}$  is outer, we have  $\# \mathfrak{F} = [\mathfrak{F}R_r : R_r]_r = [V_R(S)_i \mathfrak{F}R_r : R_r]_r = \# \mathfrak{F} \cdot [V_R(S) : V_R(R)]$  (Lemma 1), so that,  $[R : S]_i = \# \mathfrak{F} = (\mathfrak{H}' : \mathfrak{N}_0)$  and  $V_R(S)$  coincides with the center of  $R$ . Noting that  $R$  is a completely reducible  $S$ - $R$ -module,  $R$  is  $S$ - $R$ -irreducible. This completes the proofs of (1)–(2). Now, from (1)–(2) and Lemma 5, it follows that  $[\mathfrak{H}'|\mathfrak{N}_0]_r = (\mathfrak{H}' : \mathfrak{N}_0) = [R : S]_i = [\mathfrak{F}(S)|\mathfrak{N}_0]_r$ . Since  $\mathfrak{H}' \subset \mathfrak{F}(S)$ , We have  $\mathfrak{H}' = \mathfrak{F}(S)$  (Lemma 3). Hence  $V_A(S) = I(\mathfrak{H}')$  and, is a simple ring. Therefore, by Lemma 5,  $A$  is homogeneous completely reducible as  $S$ - $A$ -module, and so,  $A$  is  $S \cdot I(\mathfrak{H}')$ - $A$ -irreducible. Thus  $\mathfrak{H}'$  is an  $M$ -group.

Next, we shall present a generalization of a theorem w. r. t. centralizers in central simple algebras, which is a direct consequence of Th. 4 and Remark 2.

**Theorem 8.** *Let  $B_1 \subset B_2$  be  $M$ -subrings of  $A$  such that  $V_A^2(B_1) = B_1$  and  $[B_2 : B_1]_i \cdot [B_2 : B_1]_r < \infty$ . Then  $B_1$ 's are all simple, and there exists a 1–1 dual correspondence between simple intermediate rings of  $B_2/B_1$  and simple intermediate rings of  $V_A(B_1)/V_A(B_2)$ , in the sense of the following*

$$B = V_A(I) \quad \Leftrightarrow \quad I = V_A(B).$$

**Corollary 4.** *Let  $\mathfrak{H}_1 \subset \mathfrak{H}_2$  be  $M$ -groups such that  $\mathfrak{F}(J(\mathfrak{H}_1)) = \mathfrak{H}_1$  and*

$[\mathfrak{H}_2 || \mathfrak{H}_1]_r [I(\mathfrak{H}_2) : I(\mathfrak{H}_1)]_i < \infty$  (which is equivalent to  $[\mathfrak{H}_2 || \mathfrak{H}_1]_r [\mathfrak{H}_2 || \mathfrak{H}_1]_i < \infty$ .) Let  $\mathfrak{H}$  be an  $N$ -regular intermediate group of  $\mathfrak{H}_2/\mathfrak{H}_1$ . Then  $V_A(I(\mathfrak{H}))$  is a simple ring, and there exists a finite subset  $F$  of  $V_A(I(\mathfrak{H}))$  such that  $V_A(F) \cap I(\mathfrak{H}_2) = I(\mathfrak{H})$ . Let  $\{f_{ij}\}$  be a system of matrix units of  $V_A(I(\mathfrak{H}))$  such that the centralizer of  $\{f_{ij}\}$  in  $V_A(I(\mathfrak{H}))$  is a division ring. If the ring  $J(\mathfrak{H}_1)[F, \{f_{ij}\}]$  which is generated by  $F$  and  $\{f_{ij}\}$  over  $J(\mathfrak{H}_1)$  is left finite over  $J(\mathfrak{H}_1)$  then  $\mathfrak{H}$  is regular, and  $\mathfrak{F}(J(\mathfrak{H})) = \mathfrak{H}$ .

*Proof.* By Th. 5, we have  $I(\mathfrak{H}_i) = V_A^2(I(\mathfrak{H}_i))$  ( $i=1, 2$ ). Since  $I(\mathfrak{H}_i)$  is  $M$ -regular ( $i=1, 2$ ) and  $[I(\mathfrak{H}_2) : I(\mathfrak{H}_1)]_i [I(\mathfrak{H}_2) : I(\mathfrak{H}_1)]_r < \infty$ , it follows from Th. 8 that  $V_A^2(I(\mathfrak{H})) = I(\mathfrak{H})$  and  $V_A(I(\mathfrak{H}))$  is a simple ring. Hence there exists a finite subset  $F$  of  $V_A(I(\mathfrak{H}))$  such that  $V_A(F) \cap I(\mathfrak{H}_2) = I(\mathfrak{H})$ . We set  $B_i = J(\mathfrak{H}_i)$  ( $i=1, 2$ ),  $B = J(\mathfrak{H})$ ,  $B_0 = J(\mathfrak{H}_1)[F, \{f_{ij}\}]$ , and  $\mathfrak{H}_0 = \mathfrak{F}(B_0)$ . Then  $B_2 \subset B \subset B_1 \subset B_0$ , and  $\mathfrak{H}_2 \supset \mathfrak{H} \supset \mathfrak{H}_1 \supset \mathfrak{H}_0$ . Now, we shall prove that if  $[B_0 : B_1]_i < \infty$  then  $\mathfrak{F}(J(\mathfrak{H})) = \mathfrak{H}$  and, is regular. Since  $\mathfrak{H}_2 \supset \mathfrak{H} \supset \mathfrak{H}_1 \langle I(\mathfrak{H}) \rangle \supset \mathfrak{H}_0 \langle I(\mathfrak{H}) \rangle \supset \mathfrak{H}_0$ , in virtue of Th. 5 and Th. 7, it suffices to prove that  $\mathfrak{H}_0$  is an  $M$ -group,  $\mathfrak{F}(J(\mathfrak{H}_0)) = \mathfrak{H}_0$ ,  $[\mathfrak{H}_2 || \mathfrak{H}_0]_r < \infty$ ,  $\mathfrak{H}_0 \langle I(\mathfrak{H}) \rangle$  is regular, and  $I(\mathfrak{H}) = I(\mathfrak{H}_0 \langle I(\mathfrak{H}) \rangle)$ . Clearly  $\mathfrak{F}(J(\mathfrak{H}_0)) = \mathfrak{H}_0$ . Since  $I(\mathfrak{H}_0) = V_A(B_0) = V_A(B_1) = I(\mathfrak{H}_1)$  and  $A$  is  $I(\mathfrak{H}_1) \cdot B_1$ - $A$ -irreducible,  $A$  is  $I(\mathfrak{H}_0) \cdot B_0$ - $A$ -irreducible; whence  $B_0$  is an  $M$ -subring of  $A$ , and  $\mathfrak{H}_0$  is an  $M$ -group. Hence, it follows from Th. 3 and Th. 5 that  $J(\mathfrak{H}_0) = B_0$  and  $\infty > [B_0 : B_2]_i = [\mathfrak{H}_2 || \mathfrak{H}_0]_r$ . From  $V_A(I(\mathfrak{H})) \supset J(\mathfrak{H}_0 \langle I(\mathfrak{H}) \rangle) = B_0 \cap V_A(I(\mathfrak{H})) \supset B_2[F, \{f_{ij}\}]$ ,  $J(\mathfrak{H}_0 \langle I(\mathfrak{H}) \rangle)$  is a simple ring, and  $V_A(J(\mathfrak{H}_0 \langle I(\mathfrak{H}) \rangle)) = I(\mathfrak{H})$ ; whence  $\mathfrak{H}_0 \langle I(\mathfrak{H}) \rangle$  is regular.

For a subring  $B$  of  $A$ , if, for each finite subset  $F$  of  $A$ , the subring  $B[F]$  of  $A$  generated by  $F$  over  $B$  is a finitely generated left  $B$ -module then, the ring extension  $A/B$  will be called *left locally finite*.

Now, in Th. 6, if, in addition, the following condition is imposed :

(L)  $A/J(\mathfrak{H}_2)$  is left locally finite,

then the conditions of Coro. 4 are fulfilled by Prop. 5 which will be proved later. Hence, in this case, the result of Th. 6 takes the place of the next

**Corollary 4'.** Let  $\mathfrak{H}_1 \subset \mathfrak{H}_2$  be  $M$ -groups such that  $\mathfrak{F}(J(\mathfrak{H}_1)) = \mathfrak{H}_1$  and  $[\mathfrak{H}_2 || \mathfrak{H}_1]_r < \infty$ , and let the condition (L) be satisfied. Then there exists a 1—1 dual correspondence between  $N$ -regular intermediate groups of  $\mathfrak{H}_2/\mathfrak{H}_1$  and regular intermediate rings of  $J(\mathfrak{H}_1)/J(\mathfrak{H}_2)$ , in the sense of the following

$$\mathfrak{H} = \mathfrak{F}(B) \leftrightarrow B = J(\mathfrak{H}).$$

Moreover, we have the following

**Corollary 4''.** Let  $\mathfrak{H}_1$  be an  $N$ -regular group such that  $\mathfrak{F}(J(\mathfrak{H}_1)) = \mathfrak{H}_1$ , and  $\mathfrak{H}_2$  an  $M$ -group containing  $\mathfrak{H}_1$ . Let the condition (L) be satisfied.

(a) If  $[\mathfrak{H}_2 \| \mathfrak{H}_1]_r < \infty$  then  $\mathfrak{Z}(J(\mathfrak{H}_2)) = \mathfrak{H}_2$  and  $\mathfrak{H}_1$  is an  $M$ -group. Hence, there follows the same result as in that of Coro. 4'.

(b) Let  $[I(\mathfrak{H}_2) : I(\mathfrak{H}_1)]_r < \infty$  and  $\mathfrak{Z}(J(\mathfrak{H}_2)) = \mathfrak{H}_2$ . Then an intermediate  $N$ -group  $\mathfrak{H}$  of  $\mathfrak{H}_2/\mathfrak{H}_1$  with  $[\mathfrak{H} \| \mathfrak{H}_1]_r < \infty$  is  $N$ -regular if and only if  $\mathfrak{H}$  is an  $M$ -group, (particularly  $\mathfrak{H}_1$  is an  $M$ -group,) whence, in this case, there holds  $\mathfrak{Z}(J(\mathfrak{H})) = \mathfrak{H}$ .

In fact, if  $\{e_{ij}\}$  is a system of matrix units of  $A$  such that  $V_A(\{e_{ij}\})$  is a division ring then  $[J(\mathfrak{H}_1)[\{e_{ij}\}] : J(\mathfrak{H}_2)]_r = [\mathfrak{Z}(J(\mathfrak{H}_2)) \| \mathfrak{Z}(J(\mathfrak{H}_1)[\{e_{ij}\}])]_r$  (in the sense of Th. 2) and  $\mathfrak{Z}(J(\mathfrak{H}_2)) \supset \mathfrak{H}_2 \supset \mathfrak{H}_1 \supset \mathfrak{Z}(J(\mathfrak{H}_1)[\{e_{ij}\}])$ . If  $[\mathfrak{H}_2 \| \mathfrak{H}_1]_r < \infty$  then  $[J(\mathfrak{H}_1)[\{e_{ij}\}] : J(\mathfrak{H}_2)]_r < \infty$  (Th. 1), whence, by Coro. 4' and Th. 5,  $\mathfrak{H}_1$  is an  $M$ -group and  $\mathfrak{Z}(J(\mathfrak{H}_2)) = \mathfrak{H}_2$ . Our assertion (b) follows from Prop. 4, Th. 9 and Th. 10, which will be proved later.

The situation of Coro. 4'' will be contained in the next section. In Coro. 4', if, in particular,  $A$  is a division ring (or  $[\mathfrak{H}_2 \| \{1\}] < \infty$ ) then we can omit the condition (L) ([6]). In view of this point, one will be led to conjecture that the condition (L) is omitted from Coro. 4'. However, this is an open question.

## 7. Applications to simple ring extensions with local finiteness.

In this section, we shall discuss the regularity of  $N$ -regular groups of automorphisms in simple ring extensions with local finiteness whose extensions contain  $h$ -Galois extensions and  $q$ -Galois extensions ([3]—[10]). Moreover, in the study, the results of §1—6 will be well applied to obtain some sharpening of the results of [3, Lemma 5, Lemma 6, Th. 2, Coro. 5, and Th. 3] and [10, Th. 11, and Th. 13].

Throughout this section,  $B$  will be a subring of  $A$ . We abbreviate  $V = V_A(B)$ , and  $H = V_A^2(B)$ . Moreover, let  $E = \{e_{ij}\}$  be a system of matrix units of  $A$  such that  $V_A(E)$  is a division ring. For an intermediate ring  $B'$  of  $A/B$ , if  $B'$  is regular and  $[V : V_A(B')]_r < \infty$  then  $B'$  will be called *f-regular*. For a subgroup  $\mathfrak{H}$  of  $\mathfrak{Z}(B)$ , if  $\mathfrak{H}$  is regular and  $[V : I(\mathfrak{H})]_r < \infty$  then  $\mathfrak{H}$  will be called *f-regular*. Moreover, for a subring  $S$  of  $A$ , if  $S$  is an  $M$ -subring of  $A$  and  $J(\mathfrak{Z}(S)) = S$  then, the extension  $A/B$  will be called *h-Galois*.

Now, we shall make a remark on *f-regular* subrings of  $h$ -Galois extensions which is frequently used in our subsequent consideration. This was given mainly in the previous paper [7]. However, we include here the proof as an application of Th. 3.

**Remark 4.** Let  $A/B$  be  $h$ -Galois and left locally finite. Let  $B'$  be an intermediate ring of  $A/B$ . Then (a) if  $B' \subset H$  then  $B'$  is *f-regular*. (b) If

$B'$  is regular and  $[B':B]_i < \infty$  then  $B'$  is  $f$ -regular. (c) If  $B'$  is  $f$ -regular then  $B'$  is  $M$ -regular, and so,  $\mathfrak{F}(B')$  is  $M$ -regular.

The proof is as follows. (a): Clearly  $B' = \bigcup_i H_i$ , where  $H_i$  runs over all the subrings of  $B'$  finitely generated over  $B$ . Since  $B \subset H_i \subset H$ ,  $H_i$  is an  $M$ -subring of  $A$ . Hence,  $H_i$  is simple by Th. 3. Then, from  $[H_i|H_i] \leq [A|A]$ , one will easily see that  $B'$  is simple. Thus  $B'$  is  $f$ -regular. (b): Since  $B'$  is contained in the  $M$ -subring  $B'[E]$  of  $A$  with  $[B'[E]:B]_i < \infty$ , it follows from Th. 3 that  $\infty > [B'[E]:B]_i = [\mathfrak{F}(B) \parallel \mathfrak{F}(B'[E])]_r > [V:V_A(B'[E])]_r$ , whence  $B'$  is  $f$ -regular and, is  $M$ -regular. (c): As is easily seen, there exists a regular intermediate ring  $B^*$  of  $A/B$  such that  $B^* \subset B'$ ,  $[B^*:B]_i < \infty$ , and  $V_A(B^*) = V_A(B')$ . Then  $B^*$  is  $M$ -regular as in the proof of (b). From this, it follows that  $A$  is  $B' \cdot V_A(B')$ - $A$ -irreducible. Hence  $B'$  is  $M$ -regular.

Now, as in [7, p. 34], we may place the finite topology on the group  $\mathfrak{F}(B)$ . Here a basis for the neighbourhoods of  $\sigma \in \mathfrak{F}(B)$  consists of the sets  $U(\sigma, F) = \{\tau \in \mathfrak{F}(B); F|\tau = F|\sigma\}$ , where  $F$  runs over all the (non-empty) finite subsets of  $A$  (or subrings of  $A$  finitely generated over  $B$ ). Then  $\mathfrak{F}(B)$  is a topological group which is totally disconnected, and  $U(\sigma, F)$  is not only open but also closed. For a subset  $\mathfrak{S}$  of  $\mathfrak{F}(B)$ , we denote by  $\overline{\mathfrak{S}}$  the closure of  $\mathfrak{S}$  in the topological space  $\mathfrak{F}(B)$ . If  $\mathfrak{S}$  is a subgroup of  $\mathfrak{F}(B)$  then  $\overline{\mathfrak{S}}$  is a subgroup of  $\mathfrak{F}(B)$  too. Moreover, if  $S$  is a subset of  $A$  containing  $B$  then  $\mathfrak{F}(S)$  is a closed subgroup of  $\mathfrak{F}(B)$ . Hence, for every subset  $\mathfrak{S}$  of  $\mathfrak{F}(B)$ , we have that  $\overline{\mathfrak{S}} \subset \mathfrak{F}(J(\mathfrak{S}))$  and  $J(\overline{\mathfrak{S}}) = J(\mathfrak{S})$ .

Throughout the rest of this note, we set  $H_0 = J(\mathfrak{F}(B))$ . If  $V$  is simple then  $B \subset H_0 \subset H \subset A$ . Now, we shall prove the following

**Proposition 4.** *Let  $B$  be  $M$ -regular, and  $A/B$  left locally finite.*

(a) *If  $\mathfrak{H}$  is a subgroup of  $\mathfrak{F}(B)$  then  $A/J(\mathfrak{H}) \cap H$  is  $h$ -Galois and left locally finite. In particular,  $A/H$  and  $A/H_0$  are  $h$ -Galois and left locally finite.*

(b) *If  $\mathfrak{H}$  is an  $N$ -regular subgroup of  $\mathfrak{F}(B)$  with  $[V:I(\mathfrak{H})]_r < \infty$  then  $[J(\mathfrak{H}):J(\mathfrak{H}) \cap H]_i < \infty$ .*

(c) *If  $\mathfrak{H}$  is an  $f$ -regular subgroup of  $\mathfrak{F}(B)$  then  $A/J(\mathfrak{H})$  is  $h$ -Galois and left locally finite,  $\mathfrak{H}$  is  $M$ -regular, and  $\mathfrak{F}(J(\mathfrak{H})) = \overline{\mathfrak{H}}$ .*

*Proof.* (a): We set  $H^* = J(\mathfrak{H}) \cap H$ . Then  $J(\mathfrak{H} \langle V \rangle) = H^*$  and, is an  $M$ -subring of  $A$ . Hence  $A/H^*$  is  $h$ -Galois. For a finite subset  $F$  of  $A$ , we set  $B' = B[E, F]$ . Then  $B'$  is  $M$ -regular. Hence, by Th. 1, we have  $\infty > [B':B]_i \geq [\mathfrak{F}(B) \parallel \mathfrak{F}(B')]_r$ . Since  $V = I(\mathfrak{F}(B)) = I(\mathfrak{F}(H^*))$ , it follows from Lemma 4 that  $[\mathfrak{F}(H^*) \parallel \mathfrak{F}(H^*) \cap \mathfrak{F}(B')]_r < \infty$ . Noting that  $\mathfrak{F}(H^*)$  is  $M$ -regular, we have  $\infty > [J(\mathfrak{F}(H^*) \cap \mathfrak{F}(B')):J(\mathfrak{F}(H^*))]_i \geq [H^*[E, F]:H^*]_i$ .

(Th. 1). Thus  $A/H^*$  is left locally finite.

(b): Since  $\mathfrak{H}\langle V \rangle$  is  $M$ -regular, it follows from Th. 1 that  $\infty > [\mathfrak{H}\langle V \rangle \| \mathfrak{H}]_r \geq [J(\mathfrak{H}) : J(\mathfrak{H}\langle V \rangle)]_l = [J(\mathfrak{H}) : J(\mathfrak{H}) \cap H]_l$ .

(c): By Remark 4,  $J(\mathfrak{H})$  is  $M$ -regular; whence,  $\mathfrak{H}$  is  $M$ -regular and  $A/J(\mathfrak{H})$  is  $h$ -Galois. Moreover, by (a) and (b),  $A/J(\mathfrak{H})$  is left locally finite. Now, we shall prove that  $\mathfrak{F}(J(\mathfrak{H})) = \overline{\mathfrak{H}}$ . For a finite subset  $F$  of  $A$ , we set  $R = J(\mathfrak{H})[E, F]$ . Then  $J(\mathfrak{H})$  and  $R$  are  $M$ -regular, and  $\infty > [R : J(\mathfrak{H})]_l = [\mathfrak{F}(J(\mathfrak{H})) | \mathfrak{F}(R)]_r$  (Th. 3). Hence, by Coro. 3, we have  $\mathfrak{F}(J(\mathfrak{H})) = \mathfrak{F}(R) \cdot \mathfrak{H}$ ; then  $F | \mathfrak{F}(J(\mathfrak{H})) = F | \mathfrak{F}(R) \cdot \mathfrak{H} = F | \mathfrak{H}$ . This implies  $\mathfrak{F}(J(\mathfrak{H})) = \overline{\mathfrak{H}}$ .

**Proposition 5.** *Let  $B$  be  $M$ -regular, and  $A/B$  left locally finite. If  $W$  is a simple subring of  $V$  such that  $[V : W]_r < \infty$  and  $V_A^2(W) = W$  then  $[W' : W]_r = [W' : W]_l$  for every intermediate ring  $W'$  of  $V/W$ . (If  $W'$  is simple then, there holds additionally that  $V_A^2(W') = W'$  and  $W'$  is  $M$ -regular.) Moreover, if  $\mathfrak{H}$  is an  $N$ -regular subgroup of  $\mathfrak{F}(B)$  such that  $[V : I(\mathfrak{H})]_r < \infty$  and  $V_A^2(I(\mathfrak{H})) = I(\mathfrak{H})$  then  $[\mathfrak{H}' \| \mathfrak{H}]_r = [\mathfrak{H}' \| \mathfrak{H}]_l$  for every intermediate  $N$ -group  $\mathfrak{H}'$  of  $\mathfrak{F}(B)/\mathfrak{H}$ .*

*Proof.* For  $W$ , there exists a finite subset  $F$  of  $A$  such that  $V_r(F) \subset W$ . We set  $S = B[E, F]$ . Then  $V_A(S) \subset W \subset W' \subset V$ , and  $V_A(S)$  is a division ring. Let  $U$  be a subring of  $V$  containing  $V_A(S)$ , and, for a finite subset  $X$  of  $U$ , we set  $R = B[E, F, X]$ . Then, by Th. 1, we have  $\infty > [S : B]_l \geq [V_r(B) : V_r(S)]_r \geq [V_r(S)[X] : V_r(S)]_r = [V_r(S)[X] : V_r(R)]_r / [V_r(S) : V_r(R)]_r = [V_r(S)[X] : V_r(S)]_l$ . This implies  $[S : B]_l \geq [U : V_A(S)]_r = [U : V_A(S)]_l = [U : V_A(S)]_l$ . Hence  $[W' : V_A(S)] / [W : V_A(S)] = [W' : W]$ . If  $W'$  is simple then, since  $V$  and  $V_A(S)$  are  $M$ -regular, it follows from Th. 4 and Th. 8 that  $V_A^2(W') = W'$  and, is  $M$ -regular. Our last assertion follows from  $[I(\mathfrak{H}') : I(\mathfrak{H})]_r = [I(\mathfrak{H}') : I(\mathfrak{H})]_l$  and the definition of reduced index.

The following theorem plays important roles in our study.

**Theorem 9.** *Let  $B$  be  $M$ -regular, and  $A/B$  left locally finite. Let  $\mathfrak{H}$  be a closed  $f$ -regular subgroup of  $\mathfrak{F}(B)$ . If  $\mathfrak{H}'$  is an  $N$ -regular subgroup of  $\mathfrak{F}(B)$  containing  $\mathfrak{H}$  such that  $[\mathfrak{H}' \| \mathfrak{H}]_r < \infty$  then  $\mathfrak{H}'$  is closed and  $f$ -regular.*

*Proof.* By Prop. 4, we have  $\mathfrak{F}(J(\mathfrak{H})) = \mathfrak{H}$ . Clearly  $\mathfrak{H}'\langle V \rangle \supset \mathfrak{H}' \supset \mathfrak{H}$ , and  $\mathfrak{H}'\langle V \rangle$  is  $f$ -regular. Hence  $\mathfrak{H}'\langle V \rangle$  and  $\mathfrak{H}$  are  $M$ -regular (Prop. 4). Since  $[\mathfrak{H}'\langle V \rangle \| \mathfrak{H}]_r = [\mathfrak{H}'\langle V \rangle \| \mathfrak{H}]_l < \infty$  (Prop. 5) and  $A/J(\mathfrak{H})$  is left locally finite (Prop. 4), it follows from Coro. 4 and Prop. 4 that  $\mathfrak{H}'$  is regular and  $\mathfrak{H}' = \mathfrak{F}(J(\mathfrak{H}')) = \overline{\mathfrak{H}'}$ .

Now, for a group  $\mathfrak{H}$  of (ring) automorphisms in  $A$ , if  $V_A(J(\mathfrak{H}))$  is simple and  $\langle V_A(J(\mathfrak{H})) \rangle \subset \mathfrak{H}$  then  $\mathfrak{H}$  will be called  $(*)$ -regular. Clearly, an

$N$ -regular group  $\mathfrak{H}$  with  $I(\mathfrak{H})=V_A(J(\mathfrak{H}))$  is  $(*)$ -regular, and conversely. Particularly, every regular group is  $(*)$ -regular. For a  $(*)$ -regular subgroup  $\mathfrak{H}$  of  $\mathfrak{S}(B)$ , if  $[V:I(\mathfrak{H})]_r < \infty$  then  $\mathfrak{H}$  will be called  $(*)_f$ -regular.

The following theorem contains the results of [3, Th. 3] and [10, Th. 11 (a)].

**Theorem 10.** *Let  $B$  be  $M$ -regular, and  $A/B$  left locally finite. Let  $\mathfrak{H}$  be a  $(*)_f$ -regular subgroup of  $\mathfrak{S}(B)$ . Then*

(a)  $\mathfrak{H}$  is  $f$ -regular.

(b) *If  $W$  is a simple subring of  $V$  containing  $I(\mathfrak{H})$  such that  $W\mathfrak{H} \subset W$ , then  $\mathfrak{H}\langle W \rangle$  is  $f$ -regular, and  $\overline{\mathfrak{H}\langle W \rangle} = \overline{\mathfrak{H}}\langle W \rangle$ .*

*Proof.* We set  $\mathfrak{H}' = \mathfrak{S}(J(\mathfrak{H}))$  and  $\mathfrak{H}_0 = \mathfrak{S}(J(\mathfrak{H})[E])$ . Then  $\mathfrak{H}' \supset \mathfrak{H}_0$ , and  $\mathfrak{S}(J(\mathfrak{H}_0)) = \mathfrak{H}_0$ . Moreover,  $\mathfrak{H}'$  is a  $(*)_f$ -regular subgroup of  $\mathfrak{S}(B)$  with  $I(\mathfrak{H}') = I(\mathfrak{H})$ , and  $\mathfrak{H}_0$  is a closed regular subgroup of  $\mathfrak{S}(B)$ . Since  $\infty > [J(\mathfrak{H})[E]: H \cap J(\mathfrak{H})]_i = [\mathfrak{S}(H \cap J(\mathfrak{H})) | \mathfrak{S}(J(\mathfrak{H})[E])]_r \geq [\mathfrak{S}(J(\mathfrak{H})) | \mathfrak{H}_0]_r = [\mathfrak{H}' | \mathfrak{H}_0]_r$  (Prop. 4, and Th. 2),  $\mathfrak{H}_0$  is  $f$ -regular. Hence  $\mathfrak{H}'$  is  $f$ -regular by Th. 9. Thus  $J(\mathfrak{H}) = J(\mathfrak{H}')$  is a simple ring. Therefore  $\mathfrak{H}$  is  $f$ -regular. Next, we shall prove our second assertion (b). From  $W\mathfrak{H} \subset W$ , we have  $W\overline{\mathfrak{H}} \subset W$ . Clearly  $[\overline{\mathfrak{H}}\langle W \rangle | \overline{\mathfrak{H}}]_r < \infty$ , and  $\overline{\mathfrak{H}}\langle W \rangle$  is  $N$ -regular. Hence, by Th. 9,  $\overline{\mathfrak{H}}\langle W \rangle$  is  $f$ -regular and closed. Since  $J(\overline{\mathfrak{H}}\langle W \rangle) = J(\mathfrak{H}\langle W \rangle)$  is simple and  $W = I(\overline{\mathfrak{H}}\langle W \rangle) = I(\mathfrak{H}\langle W \rangle)$ ,  $\mathfrak{H}\langle W \rangle$  is  $f$ -regular.

The following theorem contains a sharpening of the results of [3, Th. 2] and [10, Th. 13].

**Theorem 11.** *Let  $B$  be  $M$ -regular, and  $A/B$  left locally finite. Let  $\mathfrak{H}$  be an  $N$ -regular subgroup of  $\mathfrak{S}(B)$ . Then the following conditions are equivalent.*

(1)  $\mathfrak{H}$  is  $f$ -regular.

(2)  $I(\overline{\mathfrak{H}}) = I(\mathfrak{H})$ , and  $\mathfrak{H}$  contains a  $(*)_f$ -regular subgroup of  $\mathfrak{S}(B)$  which is of right-finite reduced index in  $\mathfrak{H}$ .

(3)  $I(\overline{\mathfrak{H}}) = I(\mathfrak{H})$ ,  $V_A^2(I(\mathfrak{H})) = I(\mathfrak{G})$ , and  $[\mathfrak{H}\langle V \rangle | (\mathfrak{H} \cap \mathfrak{H}_i)\langle I(\mathfrak{H}) \rangle]_r < \infty$  for every open subgroup  $\mathfrak{H}_i$  of  $\mathfrak{S}(B)$ .

*Proof.* (1)  $\Rightarrow$  (3): Let  $\mathfrak{H}$  be  $f$ -regular. Then, it is clear that  $I(\overline{\mathfrak{H}}) = I(\mathfrak{H})$  and  $V_A^2(I(\mathfrak{H})) = I(\mathfrak{H})$ . Let  $\mathfrak{H}_i$  be an open subgroup of  $\mathfrak{S}(B)$ . Then, there exists a finite subset  $F$  of  $A$  such that  $\mathfrak{S}(B[F]) \subset \mathfrak{H}_i$ . Then  $\infty > [J(\mathfrak{H})[E, F]: J(\mathfrak{H})]_i = [\overline{\mathfrak{H}} | \mathfrak{S}(J(\mathfrak{H})[E, F])]_r$  (Prop. 4, and Th. 3). Hence, by Coro. 3, we have  $[\mathfrak{H} | \mathfrak{S}(J(\mathfrak{H})[E, F]) \cap \mathfrak{H}]_r < \infty$ . Since  $\mathfrak{S}(J(\mathfrak{H})[E, F]) \subset \mathfrak{S}(J(\mathfrak{H})) \cap \mathfrak{S}(B[F]) \subset \overline{\mathfrak{H}} \cap \mathfrak{H}_i$ , it follows that  $\mathfrak{S}(J(\mathfrak{H})[E, F]) \cap \mathfrak{H} \subset \overline{\mathfrak{H}} \cap \mathfrak{H}_i \cap \mathfrak{H} = \mathfrak{H}_i \cap \mathfrak{H}$ . Hence  $[\mathfrak{H} | (\mathfrak{H} \cap \mathfrak{H}_i)\langle I(\mathfrak{H}) \rangle]_r < \infty$ . Therefore

$$[\mathfrak{H}\langle V \rangle \| (\mathfrak{H} \cap \mathfrak{H}_i) \langle I(\mathfrak{H}) \rangle]_r = [\mathfrak{H}\langle V \rangle \| \mathfrak{H}]_r [\mathfrak{H} \| (\mathfrak{H} \cap \mathfrak{H}_i) \langle I(\mathfrak{H}) \rangle]_r = [V : I(\mathfrak{H})]_r \cdot [\mathfrak{H} \| (\mathfrak{H} \cap \mathfrak{H}_i) \langle I(\mathfrak{H}) \rangle]_r < \infty.$$

(3)  $\Rightarrow$  (2): Clearly  $[V : I(\mathfrak{H})]_r < \infty$ . Hence, from  $V_A^2(I(\mathfrak{H})) = I(\mathfrak{H})$ , there exists a finite subset  $F$  of  $V_A(I(\mathfrak{H}))$  such that  $V_r(F) = I(\mathfrak{H})$ . Since  $\mathfrak{H} \cap \mathfrak{F}(B[F]) \supset \langle I(\mathfrak{H}) \rangle$  and  $J(\mathfrak{H} \cap \mathfrak{F}(B[F])) \supset F$ ,  $\mathfrak{H} \cap \mathfrak{F}(B[F])$  is a  $(*)_f$ -regular subgroup of  $\mathfrak{F}(B)$  such that  $I(\mathfrak{H} \cap \mathfrak{F}(B[F])) = I(\mathfrak{H})$ . Clearly  $\mathfrak{F}(B[F])$  is an open subgroup of  $\mathfrak{F}(B)$ . Hence  $[\mathfrak{H} \| \mathfrak{H} \cap \mathfrak{F}(B[F])]_r < \infty$ .

(2)  $\Rightarrow$  (1): From  $I(\mathfrak{H}) = I(\mathfrak{H})$ ,  $\mathfrak{H}$  is an  $N$ -regular group. Let  $\mathfrak{H}_0$  be a  $(*)_f$ -regular subgroup of  $\mathfrak{F}(B)$  such that  $\mathfrak{H}_0 \subset \mathfrak{H}$  and  $[\mathfrak{H} \| \mathfrak{H}_0]_r < \infty$ . Then  $\mathfrak{H}_0$  is  $f$ -regular by Th. 10. Since  $\mathfrak{H}\langle V \rangle$  is  $f$ -regular, it follows from Prop. 4 and Th. 2 that  $\infty > [\mathfrak{H}\langle V \rangle \| \mathfrak{H}_0]_r \geq [J(\mathfrak{H}_0) : J(\mathfrak{H}\langle V \rangle)]_r = [\mathfrak{F}(J(\mathfrak{H}\langle V \rangle)) \| \mathfrak{F}(J(\mathfrak{H}_0))]_r = [\overline{\mathfrak{H}\langle V \rangle} \| \mathfrak{H}_0]_r \geq [\mathfrak{H} \| \mathfrak{H}_0]_r$ . Hence  $\mathfrak{H}$  is  $f$ -regular by Th. 9. Then, noting  $I(\mathfrak{H}) = I(\mathfrak{H})$ ,  $\mathfrak{H}$  is  $f$ -regular.

The following corollary is a direct consequence of Th. 9 and Th. 11 (2  $\Rightarrow$  1).

**Corollary 5.** *Let  $B$  be  $M$ -regular, and  $A/B$  left locally finite. Let  $\mathfrak{H} \subset \mathfrak{H}'$  be  $N$ -regular subgroups of  $\mathfrak{F}(B)$  such that  $[\mathfrak{H}' \| \mathfrak{H}]_r < \infty$ . If  $\mathfrak{H}$  is  $f$ -regular and if  $\mathfrak{H} = \mathfrak{H}$  or  $\mathfrak{H}' = \mathfrak{H}'$  then  $\mathfrak{H}'$  is  $f$ -regular (and closed).*

If  $V$  is finite over the center of  $A$  then there exists a finite subset  $F$  of  $A$  such that  $V_r(F)$  coincides with the center of  $A$ , and moreover, every regular (resp.  $(*)$ -regular) subgroup of  $\mathfrak{F}(B)$  is  $f$ -regular (resp.  $(*)_f$ -regular). For this case, we have the following

**Corollary 6.** *Let  $B$  be  $M$ -regular, and  $A/B$  left locally finite. Let  $V$  be finite over the center of  $A$ , and  $F$  a finite subset of  $A$  such that  $V_r(F)$  coincides with the center of  $A$ . Then every subgroup of  $\mathfrak{F}(B[F])$  is  $(*)_f$ -regular, and, for an  $N$ -regular subgroup  $\mathfrak{H}$  of  $\mathfrak{F}(B)$ , the following conditions are equivalent.*

- (1)  $\mathfrak{H}$  is regular.
- (2)  $I(\mathfrak{H}) = I(\mathfrak{H})$ , and  $[\mathfrak{H} \| \mathfrak{H} \cap \mathfrak{F}(B[F])]_r < \infty$ .
- (3)  $I(\mathfrak{H}) = I(\mathfrak{H})$ , and  $(\mathfrak{H} : (\mathfrak{H} \cap \mathfrak{H}_i) \langle I(\mathfrak{H}) \rangle) < \infty$  for every open subgroup  $\mathfrak{H}_i$  of  $\mathfrak{F}(B)$ .

*Proof.* Our first assertion is obvious.

(1)  $\Rightarrow$  (3): By Th. 11 (1  $\Rightarrow$  3).

(3)  $\Rightarrow$  (2): By the fact that  $\mathfrak{F}(B[F])$  is an open subgroup of  $\mathfrak{F}(B)$ .

(2)  $\Rightarrow$  (1): By Th. 11 (2  $\Rightarrow$  1).

**Remark 5.** Let  $B$  be  $M$ -regular, and  $A/B$  left locally finite. For

an  $N$ -regular subgroup  $\mathfrak{H}$  of  $\mathfrak{F}(B)$ , we consider the following conditions as in Th. 11.

$$(a) \quad I(\overline{\mathfrak{H}}) = I(\mathfrak{H}).$$

(b)  $\mathfrak{H}$  contains a  $(*)_f$ -regular subgroup of  $\mathfrak{F}(B)$  which is of right-finite reduced index in  $\mathfrak{H}$ .

(c)  $V_A^2(I(\mathfrak{H})) = I(\mathfrak{H})$ , and  $[\mathfrak{H}\langle V \rangle \| (\mathfrak{H} \cap \mathfrak{H}_i)\langle I(\mathfrak{H}) \rangle]_r < \infty$  for every open subgroup  $\mathfrak{H}_i$  of  $\mathfrak{F}(B)$ . (If  $V$  is finite over the center of  $A$  then the condition  $V_A^2(I(\mathfrak{H})) = I(\mathfrak{H})$  may be omitted.)

First, we shall show that conditions (b), (c) are equivalent.

(c)  $\Rightarrow$  (b): The proof proceeds as in that of Th. 11 (3  $\Rightarrow$  2), and it may be omitted.

(b)  $\Rightarrow$  (c): Let  $\mathfrak{H}_0$  be a  $(*)_f$ -regular subgroup of  $\mathfrak{F}(B)$  such that  $\mathfrak{H}_0 \subset \mathfrak{H}$  and  $[\mathfrak{H} \| \mathfrak{H}_0]_r < \infty$ . Then  $\mathfrak{H}_0$  is  $f$ -regular (Th. 10), and  $V_A^2(I(\mathfrak{H}_0)) = I(\mathfrak{H}_0) \subset I(\mathfrak{H}) \subset V$ . Since  $\mathfrak{F}(\mathfrak{H})$  is simple, we have  $V_A^2(I(\mathfrak{H})) = I(\mathfrak{H})$  (Prop. 5). Let  $\mathfrak{H}_i$  be an open subgroup of  $\mathfrak{F}(B)$ . Then, by Th. 11 (1  $\Rightarrow$  3), there holds that

$$[\mathfrak{H}_0\langle V \rangle \| (\mathfrak{H}_0 \cap \mathfrak{H}_i)\langle I(\mathfrak{H}_0) \rangle]_r < \infty.$$

Then

$$\begin{aligned} [\mathfrak{H}\langle V \rangle \| (\mathfrak{H} \cap \mathfrak{H}_i)\langle I(\mathfrak{H}) \rangle]_r &\leq [\mathfrak{H}\langle V \rangle \| (\mathfrak{H}_0 \cap \mathfrak{H}_i)\langle I(\mathfrak{H}_0) \rangle]_r \\ &= [\mathfrak{H}\langle V \rangle \| \mathfrak{H}_0\langle V \rangle]_r [\mathfrak{H}_0\langle V \rangle \| (\mathfrak{H}_0 \cap \mathfrak{H}_i)\langle I(\mathfrak{H}_0) \rangle]_r \\ &\leq [\mathfrak{H} \| \mathfrak{H}_0]_r [\mathfrak{H}_0\langle V \rangle \| (\mathfrak{H}_0 \cap \mathfrak{H}_i)\langle I(\mathfrak{H}_0) \rangle]_r < \infty. \end{aligned}$$

Secondly, we shall present an example which implies that conditions (a), (b) are independent to each other.

**Example.** In pp. 23—24 of [2], G. Köthe proved that there exist a (countably) infinite number of central and cyclic division algebras over the rational number field  $Q$  of which the degrees are prime to each other:  $\{K_0, K_1, K_2, \dots\}$ . Then each  $K_i$  contains a maximal subfield  $M_i$  which is a cyclic extension over  $Q$  with the cyclic Galois group  $[\sigma_i]$  generated by  $\sigma_i$ , where  $\sigma_i = M_i | \langle a_i \rangle$  for some element  $a_i$  of  $K_i$ . We may suppose that  $n = [M_0 : Q]$  is odd, and  $a_0^n$  is a positive number of  $Q$ . As is well known,  $K_0 \otimes_Q K_1 \otimes \dots \otimes_Q K_s$  (tensor product) is a division ring. Hence, the following limit

$$K = \lim_{\substack{\longrightarrow \\ s}} K_0 \otimes_Q M_1 \otimes_Q \dots \otimes_Q M_s$$

is a division ring. The center of  $K$  is  $C = \lim M_1 \otimes \dots \otimes M_s$ . Clearly  $K/Q$  is locally finite,  $h$ -Galois, and  $[V_K(Q) : C] = n^2$ . As is easily seen,  $\mathfrak{F}(Q)$  (the group of (ring) automorphisms in  $K$ ) contains the following automorphisms

$$\sigma = \lim 1_0 \otimes \sigma_1 \otimes \cdots \otimes \sigma_s$$

$$\sigma_i' = \lim 1_0 \otimes 1_1 \otimes \cdots \otimes 1_{i-1} \otimes \sigma_i \otimes 1_{i+1} \otimes \cdots \otimes 1_s \quad (i \geq 1),$$

where  $1_j$ 's are identity maps. Now, under this situation, we shall present two types of  $N$ -regular subgroups of  $\mathfrak{F}(Q)$  such that one is with (a) but not with (b), and the other one is with (b) but not with (a).

(i) Let  $\mathfrak{H}$  be the cyclic group generated by  $\langle a_0 + 1 \rangle \cdot \sigma$ . Then, as in [6, Example],  $\mathfrak{H}$  is an  $N$ -regular subgroup of  $\mathfrak{F}(Q)$  which is closed but not regular. Hence  $\mathfrak{H}$  satisfies (a) but not satisfies (b).

(ii) We set

$$\tau = \langle a_0 \rangle \cdot \sigma, \quad \mathfrak{T} = \bigcup_{s=1}^{\infty} [\sigma_1'] \cdot \cdots \cdot [\sigma_s'],$$

$$\mathfrak{H}_1 = [\tau^n] \cdot \mathfrak{T}, \text{ and } \mathfrak{H}_2 = [\tau] \cdot \mathfrak{T}.$$

Then, since  $\mathfrak{T}$  is a group of outer automorphisms of finite orders which commutes with outer automorphisms  $\tau^m (m \neq 0)$  of infinite orders, it follows that  $\mathfrak{H}_1 \subset \mathfrak{H}_2$  are groups,  $(\mathfrak{H}_2 : \mathfrak{H}_1) = n < \infty$ ,  $I(\mathfrak{H}_1) = I(\mathfrak{H}_2) = C$ ,  $\mathfrak{H}_1$  is regular, and  $\mathfrak{H}_2$  is  $N$ -regular; however,  $\mathfrak{H}_2 \supset \mathfrak{T} \ni \sigma$ , and whence  $\mathfrak{H}_2 \ni \tau \sigma^{-1} = \langle a_0 \rangle$ ; thus, we obtain  $\overline{I}(\mathfrak{H}_2) \cong C = I(\mathfrak{H}_2)$ .

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