ON N-GROUPS OF AUTOMORPHISMS IN DIVISION RINGS

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Throughout the present paper, A will be a division ring, and $\mathfrak{G}$ the group of all automorphisms of A. For any element a of A, we denote by a_l (resp. a_r) the left (resp. right) multiplication determined by a, and for any non-zero element a of A, we denote by $\langle a \rangle$ the inner automorphism $a_l \cdot a_r^{-1}$ of A. Moreover, for any subset E of A and for any subset $\mathfrak{S}$ of $\mathfrak{G}$, we shall use the following conventions: E_l (resp. E_r) = the set of left (resp. right) multiplications determined by elements of E; $\langle E \rangle$ = the set of inner automorphisms determined by non-zero elements of E; $V_A(E)$ = the centralizer of E in A; $\mathfrak{G}(E)$ = the set of E-ring automorphisms of A; $I(\mathfrak{S})$ = the subring of A generated by all the non-zero elements inducing inner automorphisms belonging to $\mathfrak{S}$; $J(\mathfrak{S})$ = the set of $\mathfrak{S}$-invariant elements of A; $V(\mathfrak{S}) = V_A(J(\mathfrak{S}))$. Let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$. If $\langle I(\mathfrak{H}) \rangle \subset \mathfrak{H}$ then $I(\mathfrak{H})$ is a division ring, and $\mathfrak{H}$ will be called an *N-group* (cf. [1, Def. VI. 8. 2]). Moreover, if $\mathfrak{G}(J(\mathfrak{H})) = \mathfrak{H}$ then $\mathfrak{H}$ will be called a *T-group*. Clearly a T-group is an N-group. In general, for any subset E of A, $\mathfrak{G}(E)$ is a T-group, and so, an N-group. If $\mathfrak{H}$ is an N-group of finite reduced order (cf. [1, Def. VI. 10. 1]) then $\mathfrak{H}$ is a T-group as is well known in finite Galois theory.

In this paper, the main theme of our discussion will concern the relation of N-groups to T-groups, which contains a characterization of T-groups. Moreover, for N-groups, we shall present a generalization of the notion of reduced orders, which plays impiortant rôles in our study. This will be given later under the calling of reduced indices.

For a subset E of A and for a subset $\mathfrak{S}$ of $\mathfrak{G}$, we denote by $\mathfrak{S}|E$ the restriction of $\mathfrak{S}$ to E; if, for a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ and for σ, $\tau \in \mathfrak{S}$, $\sigma|E = (\tau|E)\delta$ (: $E \xrightarrow{\tau} A \xrightarrow{\delta} A$) for some δ in $\mathfrak{H}$, then we write $\sigma|E \sim \tau|E$ (mod $\mathfrak{H}$). Evidently, the relation $\sim$ is an equivalence relation in $\mathfrak{S}|E$, and the cardinal number of the equivalence classess w. r. t. $\sim$ is denoted as $(\mathfrak{S}|E : \mathfrak{H})$. For any subgroups $\mathfrak{H}_1$, $\mathfrak{H}_2$ of $\mathfrak{G}$ such that $\mathfrak{H}_1 \subset \mathfrak{H}_2$, we denote by $(\mathfrak{H}_2 : \mathfrak{H}_1)$ the index of $\mathfrak{H}_1$ in $\mathfrak{H}_2$. For a division ring D and for a left (resp. right) D-module M, we denote by $[M : D]_l$ (resp. $[M : D]_r$) the left (resp. right) dimension of M. In case $[M : D]_l = [M : D]_r$ they are denoted by $[M : D]$. As to other notations and terminologies used in this paper, we follow [4].

Throughout the rest of this paper, $\mathfrak{H}$, $\mathfrak{H}_1$, $\mathfrak{H}_2$, etc. will be subgroups of $\mathfrak{G}$. Now we shall begin our study with the following lemma which is a direct consequence of [7, Lemmas 1.3—1.5] and [1, Density theorem for irreducible modules, p. 31].

Lemma 1. *Let D be a division subring of A. Then*

(a) *if D' is a division subring of D then*
$$[D: D']_l \geqslant [(\mathfrak{G}(D')|D)A_r: A_r]_r$$
$$= (\mathfrak{G}(D')|D: \langle V_A(D')\rangle)[V_A(D'): V_A(D)]_r,$$

(b) *if $\mathfrak{H}$ is a group such that $J(\mathfrak{H}) \subset D$ then*
$$[D: J(\mathfrak{H})]_l = [(\mathfrak{H}|D)A_r: A_r]_r,$$

provided we do not distinguish between two infinite dimensions.

Corollary 1. *If $\mathfrak{H}_1 \subset \mathfrak{H}_2$ and $(\mathfrak{H}_2: \mathfrak{H}_1) < \infty$ then $[J(\mathfrak{H}_1): J(\mathfrak{H}_2)] \leqslant (\mathfrak{H}_2: \mathfrak{H}_1)$.*

Proof. We set $B_i = J(\mathfrak{H}_i)\,(i=1, 2)$, and $\mathfrak{H}_2 = \mathfrak{H}_1\sigma_1 \cup \cdots \cup \mathfrak{H}_1\sigma_n$, where $n = (\mathfrak{H}_2: \mathfrak{H}_1)$. Then, by Lemma 1 (b), we have $[B_1: B_2]_l = [(\mathfrak{H}_2|B_1)A_r: A_r]_r \leqslant n = (\mathfrak{H}_2: \mathfrak{H}_1)$. For a finite subset E of B_1 which generates B_1 over B_2, we set $B' = J(\mathfrak{G}(E\,\mathfrak{H}_2) \cap \mathfrak{H}_2)$. Then B' is a division subring of A containing B_1. Since $\mathfrak{G}(E\,\mathfrak{H}_2) \cap \mathfrak{H}_2$ is a normal subgroup of $\mathfrak{H}_2$ which is of finite index in $\mathfrak{H}_2$, $\mathfrak{H}_2|B'$ is a finite group of automorphisms in B' such that the set of $(\mathfrak{H}_2|B')$-invariant elements of B' is B_2. If we take B' instead of A, then by Lemma 1, we obtain $\infty > [B': B_2]_l \geqslant [V_{B'}(B_2): V_{B'}(B')] \geqslant [V_{B'}(B_2): V_{B'}(B_1)]$, and whence $[B_1: B_2]_l = [(\mathfrak{G}(B_2, B')|B_1)A_r: A_r]_r = (\mathfrak{G}(B_2, B')|B_1: \langle V_{B'}(B_2)\rangle)\cdot[V_{B'}(B_2): V_{B'}(B_1)] = [B_1: B_2]_r$, where $\mathfrak{G}(B_2, B')$ is the group of all B_2-ring automorphisms in B'.

Lemma 2. *Let S be a subset of A, $\mathfrak{S} = \mathfrak{G}(S)$, and $\mathfrak{S} \subset \mathfrak{H}$. If $\mathfrak{N}$ is a normal subgroup of $\mathfrak{H}$ then $(\mathfrak{H}|S: \mathfrak{N}) = (\mathfrak{H}: \mathfrak{S}\mathfrak{N})$.*

Proof. Let σ, $\tau \in \mathfrak{H}$. Since $\mathfrak{S}\mathfrak{N}\sigma = \mathfrak{S}\sigma\mathfrak{N}$, we have $\mathfrak{S}\mathfrak{N}\sigma|S = \mathfrak{S}\sigma\mathfrak{N}|S = (\sigma|S)\mathfrak{N}$. Hence, if $\mathfrak{S}\mathfrak{N}\sigma = \mathfrak{S}\mathfrak{N}\tau$ then $(\sigma|S)\mathfrak{N} = \mathfrak{S}\mathfrak{N}\sigma|S = \mathfrak{S}\mathfrak{N}\tau|S = (\tau|S)\mathfrak{N}$. This implies $\sigma|S \sim \tau|S$ (mod $\mathfrak{N}$), Next, suppose $\sigma|S \sim \tau|S$ (mod $\mathfrak{N}$). Then $\sigma|S = (\tau|S)\delta$ for some $\delta \in \mathfrak{N}$, and this is equal to $\tau\delta|S$. Hence there exists an element $\varepsilon \in \mathfrak{G}(S) = \mathfrak{S}$ such that $\sigma = \varepsilon(\tau\delta) = \varepsilon(\tau\delta\tau^{-1})\tau$ and $\varepsilon(\tau\delta\tau^{-1}) \in \mathfrak{S}\mathfrak{N}$. Thus we obtain $\mathfrak{S}\mathfrak{N}\sigma = \mathfrak{S}\mathfrak{N}\tau$. We have therefore proved that $(\mathfrak{H}: \mathfrak{S}\mathfrak{N}) = (\mathfrak{H}|S: \mathfrak{N})$.

Definition. For N-groups $\mathfrak{H}_1$, $\mathfrak{H}_2$ such that $\mathfrak{H}_1 \subset \mathfrak{H}_2$, we denote by $(\mathfrak{H}_2 \| \mathfrak{H}_1)_r$ (resp. $(\mathfrak{H}_2 \| \mathfrak{H}_1)_l$) the product $(\mathfrak{H}_2: \mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle)[I(\mathfrak{H}_2): I(\mathfrak{H}_1)]_r$ (resp. $(\mathfrak{H}_2: \mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle)[I(\mathfrak{H}_2): I(\mathfrak{H}_1)]_l$), which is called the *right* (resp. *left*) *reduced index* of $\mathfrak{H}_1$ in $\mathfrak{H}_2$. In case $(\mathfrak{H}_2 \| \mathfrak{H}_1)_r = (\mathfrak{H}_2 \| \mathfrak{H}_1)_l$ they are denoted by $(\mathfrak{H}_2 \| \mathfrak{H}_1)$.

Clearly $(\mathfrak{H}_2 \| \{1\})$ is the reduced order of $\mathfrak{H}_2$, which has been introduced in [1]. If $I(\mathfrak{H}_2)$ is finite over its center then $[I(\mathfrak{H}_2): I(\mathfrak{G}_1)]_r =$

$[I(\mathfrak{H}_2): I(\mathfrak{H}_1)]_l$ (cf. [1, Prop. VII. 1. 3]), and whence, in this case, we have $(\mathfrak{H}_2 \| \mathfrak{H}_1)_r = (\mathfrak{H}_2 \| \mathfrak{H}_1)_l$.

For reducd indices, we shall prove the following

Lemma 3. *Let $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ be N-groups such that $\mathfrak{H}_1 \subset \mathfrak{H}_2 \subset \mathfrak{H}_3$. Then*

(a) $(\mathfrak{H}_3 \| \mathfrak{H}_1)_r = (\mathfrak{H}_3 \| \mathfrak{H}_2)_r(\mathfrak{H}_2 \| \mathfrak{H}_1)_r$.

(b) *If $(\mathfrak{H}_2 \| \mathfrak{H}_1)_r = (\mathfrak{H}_3 \| \mathfrak{H}_1)_r < \infty$ then $\mathfrak{H}_2 = \mathfrak{H}_3$.*

Proof. (a) Since $\langle I(\mathfrak{H}_3)\rangle$ is a normal subgroup of $\mathfrak{H}_3$ and $\langle I(\mathfrak{H}_3)\rangle \cap \mathfrak{H}_2 = \langle I(\mathfrak{H}_2)\rangle$, we have

$$
\begin{aligned}
(\mathfrak{H}_3 \| \mathfrak{H}_1)_r &= (\mathfrak{H}_3 : \mathfrak{H}_1\langle I(\mathfrak{H}_3)\rangle)[I(\mathfrak{H}_3): I(\mathfrak{H}_1)]_r \\
&= (\mathfrak{H}_3: \mathfrak{H}_2\langle I(\mathfrak{H}_3)\rangle)(\mathfrak{H}_2\langle I(\mathfrak{H}_3)\rangle: \mathfrak{H}_1\langle I(\mathfrak{H}_3)\rangle)[J(\mathfrak{H}_3): I(\mathfrak{H}_1)]_r \\
&= (\mathfrak{H}_3; \mathfrak{H}_2\langle I(\mathfrak{H}_3)\rangle)(\mathfrak{H}_2: \mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle)\cdot \\
&\quad [I(\mathfrak{H}_3): I(\mathfrak{H}_2)]_r[I(\mathfrak{H}_2): I(\mathfrak{H}_1)]_r \\
&= (\mathfrak{H}_3 \| \mathfrak{H}_2)_r(\mathfrak{H}_2 \| \mathfrak{H}_1)_r
\end{aligned}
$$

(b) From $\mathfrak{H}_2 \subset \mathfrak{H}_2\langle I(\mathfrak{H}_3)\rangle \subset \mathfrak{H}_3$, we have

$$
\begin{aligned}
(\mathfrak{H}_2 \| \mathfrak{H}_1)_r &= (\mathfrak{H}_2\langle I(\mathfrak{H}_3)\rangle \| \mathfrak{H}_1)_r \\
&= (\mathfrak{H}_2\langle I(\mathfrak{H}_3)\rangle: \mathfrak{H}_1\langle I(\mathfrak{H}_3)\rangle)[I(\mathfrak{H}_3): I(\mathfrak{H}_1)]_r \\
&= (\mathfrak{H}_2: \mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle)[I(\mathfrak{H}_3): I(\mathfrak{H}_2)]_r[I(\mathfrak{H}_2): I(\mathfrak{H}_1)]_r \\
&= (\mathfrak{H}_2 \| \mathfrak{H}_1)_r[I(\mathfrak{H}_3): I(\mathfrak{H}_2)]_r < \infty.
\end{aligned}
$$

This means $[I(\mathfrak{H}_3): I(\mathfrak{H}_2)]_r = 1$, and whence $I(\mathfrak{H}_3) = I(\mathfrak{H}_2)$. Then

$$
\begin{aligned}
(\mathfrak{H}_2: \mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle) &= (\mathfrak{H}_2 \| \mathfrak{H}_1)_r / [I(\mathfrak{H}_2): I(\mathfrak{H}_1)]_r \\
&= (\mathfrak{H}_3 \| \mathfrak{H}_1)_r / [I(\mathfrak{H}_3): I(\mathfrak{H}_1)]_r \\
&= (\mathfrak{H}_3: \mathfrak{H}_1\langle I(\mathfrak{H}_3)\rangle) < \infty.
\end{aligned}
$$

Hence $\mathfrak{H}_2 = \mathfrak{H}_3$.

Now, we shall prove the following

Theorem 1. *Let B_1, B_2 be division subrings of A such that $B_1 \subset B_2$, and let $\mathfrak{H}_1$, $\mathfrak{H}_2$ be N-groups such that $\mathfrak{H}_1 \subset \mathfrak{H}_2$. Then*

(a) $[B_2: B_1]_l \geqslant (\mathfrak{G}(B_1) \| \mathfrak{G}(B_2))_r$ *(If $J(\mathfrak{G}(B_1)) = B_1$ then the equality holds.)*,

(b) $(\mathfrak{H}_2 \| \mathfrak{H}_1)_r \geqslant [J(\mathfrak{H}_1): J(\mathfrak{H}_2)]_l$ *(If $\mathfrak{G}(J(\mathfrak{H}_1)) = \mathfrak{H}_1$ then the equality holds.)*,

provided we do not distingnish between two infinite cardinal numbers.

Proof. (a, $\geqslant$) We set $V_i = V_A(B_i)$ $(i = 1, 2)$. Then

$$
\begin{aligned}
[B_2: B_1]_l &\geqslant (\mathfrak{G}(B_1) | B_2 : \langle V_1\rangle)[V_1 : V_2]_r && \text{(Lemma 1 (a))} \\
&= (\mathfrak{G}(B_1): \mathfrak{G}(B_2)\langle V_1\rangle)[V_1 : V_2]_r && \text{(Lemma 2)} \\
&= (\mathfrak{G}(B_1) \| \mathfrak{G}(B_2))_r.
\end{aligned}
$$

(b, $\geqslant$) In case $(\mathfrak{H}_2 \| \mathfrak{H}_1)_r = \infty$, our assertion is trivial. Hence we may suppose that $(\mathfrak{H}_2 \| \mathfrak{H}_1)_r < \infty$. We set $D_i = J(\mathfrak{H}_i)$ (i = 1, 2), and $E = J(\mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle)$. Then

$$
[E: D_2] \leqslant (\mathfrak{H}_2 : \mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle) \qquad \text{(Coro. 1)}
$$

and

$$[D_1: E]_l = [(\mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle | D_1)A_r: A_r]_r \quad \text{(Lemma 1 (b))}$$
$$= [(\langle I(\mathfrak{H}_2)\rangle | D_1)A_r: A_r]_r$$
$$= [(I(\mathfrak{H}_2)_l | D_1)A_r: A_r]_r$$
$$\leqslant [I(\mathfrak{H}_2): I(\mathfrak{H}_1)]_r \quad \text{(Note } D_1 \subset V_A(I(\mathfrak{H}_1))\text{)}.$$

Therefore

$$[D_1: D_2]_l = [D_1: E]_l [E: D_2]_l$$
$$\leqslant [I(\mathfrak{H}_2): I(\mathfrak{H}_1)]_r(\mathfrak{H}_2: \mathfrak{H}_1\langle I(\mathfrak{H}_2)\rangle) = (\mathfrak{H}_2 \parallel \mathfrak{H}_1)_r.$$

(The equalities of a, b) By the above inequalities, we have

$$[B_2: B_1]_l \geqslant (\mathfrak{G}(B_1) \parallel \mathfrak{G}(B_2))_r \quad \text{(a, }\geqslant\text{)}$$
$$\geqslant [J(\mathfrak{G}(B_2)): J(\mathfrak{G}(B_1))]_l \quad \text{(b, }\geqslant\text{)}.$$

Clearly $J(\mathfrak{G}(B_2)) \supset B_2$. Hence, if $J(\mathfrak{G}(B_1)) = B_1$ then the equalities hold for all. Simialrly, if $\mathfrak{G}(J(\mathfrak{H}_1)) = \mathfrak{H}_1$ then $(\mathfrak{H}_2 \parallel \mathfrak{H}_1)_r = [J(\mathfrak{H}_1): J(\mathfrak{H}_2)]_l$. This completes the proof.

As a simple corollary of Th. 2 (b), we obtain the following

Corollary 2. *Let $\mathfrak{H}_1$ be a T-group. If $\mathfrak{H}_2$ is an N-group containing $\mathfrak{H}_1$ such that $I(\mathfrak{H}_2)$ is finite over its center, then $(\mathfrak{H}_2 \parallel \mathfrak{H}_1) = [J(\mathfrak{H}_1): J(\mathfrak{H}_2)]$, provided we do not distinguish between two infinite cardinal numbers.*

Theorem 2. *Let $\mathfrak{H}_1$ be a T-group. If $\mathfrak{H}_2$ is an N-group containing $\mathfrak{H}_1$ such that the right (or left) reduced index of $\mathfrak{H}_1$ in $\mathfrak{H}_2$ is finite, then $\mathfrak{H}_2$ is a T-group.*

Proof. By Th. 1, we have $\infty > (\mathfrak{H}_2 \parallel \mathfrak{H}_1)_r = [J(\mathfrak{H}_1): J(\mathfrak{H}_2)]_l = (\mathfrak{G}(J(\mathfrak{H}_2)) \parallel \mathfrak{G}(J(\mathfrak{H}_1)))_r = (\mathfrak{G}(J(\mathfrak{H}_2)) \parallel \mathfrak{H}_1)_r$. Clearly $\mathfrak{G}(J(\mathfrak{H}_2)) \supset \mathfrak{H}_2$. Hence, by Lemma 3 (b), we obtain $\mathfrak{H}_2 = \mathfrak{G}(J(\mathfrak{H}_2))$.

Remark 1. Let B_1 be a division subring of A such that A/B_1 is Galois (i. e., $J(\mathfrak{G}(B_1)) = B_1$). If B_2 is a subring of A containing B_1 which is left (or right) finite over B_1, then B_2 is a division ring, and by Th. 1, $\infty > [B_2: B_1]_l = (\mathfrak{G}(B_1) \parallel \mathfrak{G}(B_2))_r = [J(\mathfrak{G}(B_2)): J(\mathfrak{G}(B_1))]_l = [J(\mathfrak{G}(B_2)): B_1]_l$. Clearly $J(\mathfrak{G}(B_2)) \supset B_2$. Hence $J(\mathfrak{G}(B_2)) = B_3$, that is, A/B_2 is Galois. This is the result of [6, Th. 1].

By Th. 2 and Remark 1, we obtain the following

Theorem 3. *Let $\mathfrak{H}_1$ be a T-group, and $\mathfrak{H}_2$ an N-group containing $\mathfrak{H}_1$ such that the right (or left) reduced index of $\mathfrak{H}_1$ in $\mathfrak{H}_2$ is finite. Then there exists a 1—1 dual correspondence between intermediate N-groups of $\mathfrak{H}_2/\mathfrak{H}_1$ and intermediate rings of $J(\mathfrak{H}_1)/J(\mathfrak{H}_2)$, in the following sense:*

$$\mathfrak{H}' = \mathfrak{G}(B') \longleftrightarrow B' = J(\mathfrak{H}').$$

Remark 2. In case $\mathfrak{H}_1 = \{1\}$, the result of Th. 3 is the Fundamental Theorem of finite Galois theory of division rings ([1]).

Finally, we shall present an example which implies that in Th. 2, the assuption $(\mathfrak{H}_2 \parallel \mathfrak{H}_1)_r < \infty$ plays important roles.

Example. As in [5, Example 4 (a)], we consider a dlvision ring

$$K = \lim_{\overrightarrow{s}} (\textstyle\sum_{\oplus 1}^n a^i M_0) \otimes_Q M_1 \otimes_Q \cdots \otimes_Q M_s$$

, where $\{M_s\}$ is a (countably) infinite number of normal extensions over the rational number field Q of which the degrees are prime to each other, and $K_0 = \sum_{\oplus 1}^n a^i M_0$ is a central division algebra over Q such that $a^n = c \in Q$, $M_0\langle a\rangle = M_0$, and n is odd. Clearly K/Q is locally finite, h-Galois, and the Golois group is locally compact in finite topology ([3 and 7]). If $a^n = c < 0$ then $(-a)^n = -c > 0$. Hence we may suppose $a^n = c > 0$. Then, for every natural number m, we have

$$(a+1)^m = \textstyle\sum_{0 \leq i \leq m} \binom{m}{i} a^i \notin Q .$$

Hence, for every non-zero integer x, we obtain $(a+1)^x \notin Q$, so that $\langle a + 1\rangle^x | K_0 \neq 1$. If we set $M = \lim\limits_{\rightarrow} M_1 \otimes_Q \cdots \otimes_Q M_s$ then $K = K_0 \otimes_Q M$. Since $[M_i : Q]$'s are prime to each other, we can easily find some automorphism σ of M/Q of which the order is infinite. We set $\tau = \langle a+1\rangle \otimes \sigma$, and $\mathfrak{H}$ will be the cyclic group generated by τ. If δ is an element of the closure of $\mathfrak{H}$ in finite topology, then there exists some integer t such that $\delta | K_0 = \tau^t | K_0$, and so, $\delta | K_0 = \langle a+1\rangle^t | K_0$. Moreover, for any intermediate subring K_i of K/K_0 which is finite over Q, we have $\delta | K_i = \tau^{t'} | K_i$ and so, $\delta | K_0 = \langle a+1\rangle^{t'} | K_0$. Since $\langle a+1\rangle^x | K_0 \neq 1$ for every non-zero integer x, we obtain $t = t'$. This implies that $\delta = \tau^t \in \mathfrak{H}$, Hence $\mathfrak{H}$ is closed. Clearly $\langle I(\mathfrak{H})\rangle = \{1\}$, and so, $\mathfrak{H}$ is an N-group. For an open subgroup $\mathfrak{G}(K_0)$ of $\mathfrak{G}(Q)$, we have $\mathfrak{H} \cap \mathfrak{G}(K_0) = \{1\}$. Since $(\mathfrak{H} \parallel \{1\}) = \infty$, it follows from [3, Coro. 5] that $\mathfrak{H}$ is not a T-group. Clearly $\mathfrak{H}$ contains only one T-group $\{1\}$.

REFERENCES

[1] N. JACOBSON: Structure of rings, Providence (1956).

[2] G. KÖTHE: Schiefkörper unendlichen Ranges über dem Zentrum, Math. Ann., 105 (1931), 14—39.

[3] T. NAGAHARA: On Galois conditions and Galois groups of simple rings, Trans. Amer. Math. Soc., 116 (1965), 417—434.

[4] T. NAGAHARA: Quasi-Galois extensions of simple rings, Math. J. Okayama Univ., 12 (1966), 107—146.

[5] T. NAGAHARA and H. TOMINAGA: On Galois theory of division rings, Math. J. Okayama Univ., 6 (1956), 1—21.

[6] T. NAGAHARA and H. TOMINAGA: Some remarks on Galois extensions of division rings.

Math. J. Okayama Univ., 9 (1959), 5—8.
[7] T. NAGAHARA and H. TOMINAGA: On Galois theory of simple rings, Math. J. Okayama Univ., 11 (1963), 79—117.

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Added in proof: Our lemma 1 is still true for the replacement of a division subring D of A by a subring D of A. Hence in Th. 1 (a), a division subring B_2 of A may be replaced by a subring B_2 of A.